

UNCLASSIFIED

A 218282

Armed Services Technical Information Agency

ARLINGTON HALL STATION
ARLINGTON 12 VIRGINIA

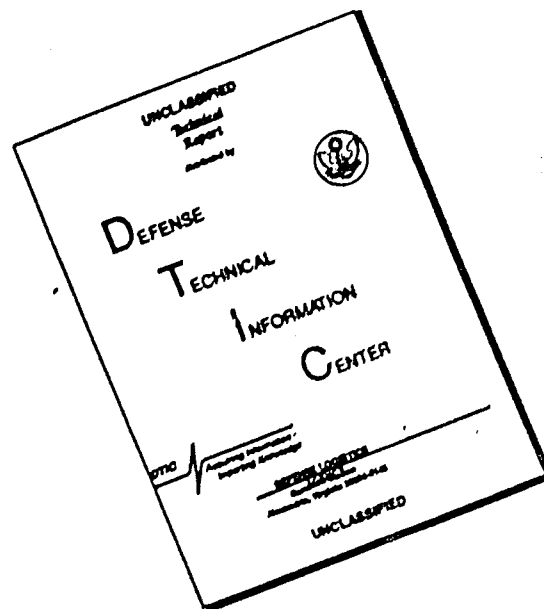
FOR
MICRO-CARD
CONTROL ONLY

1 OF 1

NOTICE: WHEN GOVERNMENT OR OTHER DRAWINGS, SPECIFICATIONS OR OTHER DATA ARE USED FOR ANY PURPOSE OTHER THAN IN CONNECTION WITH A DEFINITELY RELATED GOVERNMENT PROCUREMENT OPERATION, THE U. S. GOVERNMENT THEREBY INCURS NO RESPONSIBILITY, NOR ANY OBLIGATION WHATSOEVER; AND THE FACT THAT THE GOVERNMENT MAY HAVE FORMULATED, FURNISHED, OR IN ANY WAY SUPPLIED THE SAID DRAWINGS, SPECIFICATIONS, OR OTHER DATA IS NOT TO BE REGARDED BY IMPLICATION OR OTHERWISE, IN ANY MANNER LICENSING THE HOLDER OR ANY OTHER PERSON OR CORPORATION, OR CONVEYING ANY RIGHTS OR PERMISSION TO MANUFACTURE, USE OR SELL ANY PATENTED INVENTION THAT MAY IN ANY WAY BE RELATED THERETO.

UNCLASSIFIED

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

REPORT 60

AD No. 219282

ASTIA FILE COPY

REPORT 60

ADVISORY GROUP FOR AERONAUTICAL
RESEARCH AND DEVELOPMENT

REPORT 60

POST BUCKLING BEHAVIOUR OF STRUCTURES

by

A. VAN DER NEUT

2

FILE COPY

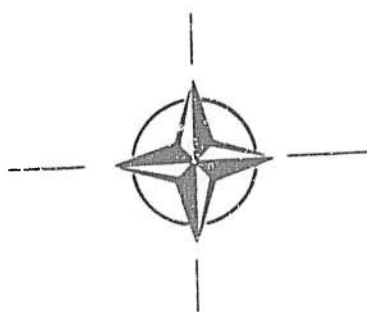
AUGUST 1956

RETURN TO

ASTIA

ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

ATTN: TISSS



NORTH ATLANTIC TREATY ORGANIZATION
PALAIS DE CHAILLOT, PARIS 16

ASTIA
RECEIVED
1956
11/10/56
11/10/56

REPORT 60

NORTH ATLANTIC TREATY ORGANIZATION
ADVISORY GROUP FOR AERONAUTICAL RESEARCH AND DEVELOPMENT

POST BUCKLING BEHAVIOUR OF STRUCTURES

by

A. van der Neut

This Report was presented at the Fourth Meeting of the Structures and Materials Panel,
held in August, 1956, at Brussels


SUMMARY

Weight efficiency when using thin plates in aircraft structures requires increasing load carrying capacity beyond buckling load. In this respect flat plates supported along their edges differ favourably from columns and curved plates.

The aim is to find the relation between the edge displacements of a panel and its load, including the relation between the increments of both buckling and stiffness.

The physical characteristic feature is that the deflections are finite, from which the equations are non-linear. Even with simple boundary conditions exact solutions are missing. Approximate solutions based upon energy theorems are available. Solutions established for flat plates at constant temperature and for thermal loading will be discussed and compared with available experimental evidence.

From a recent investigation it appears that the post buckling behaviour of narrow cylindrical panels can present the unstable explosive character, known with full cylindrical shells, at least at deformations close to those at which buckling starts.



539.384.4

2b2c3

SOMMAIRE

L'emploi de plaquettes minces demande des structures capables de résister aux charges supérieures à la charge de flambage. A cet égard les plaquettes minces supportées le long de leurs bords présentent des caractéristiques favorables vis-à-vis celles des colonnes et des plaquettes courbées.

On cherche à établir la relation entre les déplacements de bord d'un panneau de revêtement et la charge subie, y compris le rapport entre une augmentation de flambage et de rigidité.

La particularité physique est que les déflexions sont finies, donnant ainsi des équations non linéaires. Même dans des conditions de limite simples, des solutions précises ne sont pas possibles. On dispose de solutions fondées sur des théorèmes énergétiques. Des solutions établies pour des plaquettes planes à température constante soumises à des sollicitations thermiques sont étudiées et comparées avec les résultats expérimentaux disponibles. D'après une étude récente il paraît que le comportement après flambage de panneaux cylindriques étroits peuvent présenter le caractère instable explosif bien connu dans le cas de coquilles cylindriques, tout au moins lors des déformations voisines à celles provoquant le flambage.

539.384.4

2b2c3

CONTENTS

| | Page |
|--|------|
| SUMMARY | ii |
| LIST OF FIGURES | v |
| NOTATION | vi |
| 1. INTRODUCTION | 1 |
| 2. VARIETIES OF POST-BUCKLING BEHAVIOUR | 2 |
| 3. THE PHYSICAL CONCEPT OF POST BUCKLING BEHAVIOUR | 4 |
| 4. THE EQUATIONS GOVERNING POST-BUCKLING BEHAVIOUR OF FLAT PLATES | 7 |
| 5. METHODS FOR SOLUTION | 9 |
| 6. AVAILABLE SOLUTIONS FOR BEHAVIOUR AFTER BUCKLING | 11 |
| 6.1 Longitudinal Compression (Effective Width) | 11 |
| 6.1.1 Flat plates | 11 |
| 6.1.2 Narrow cylindrical panels | 13 |
| 6.2 Flat Plates Under Shear Load | 14 |
| 6.3 Flat Plates Under Thermal Stresses | 17 |
| REFERENCES | 20 |
| FIGURES | 22 |
| DISTRIBUTION | |

LIST OF FIGURES

| | Page |
|---|------|
| Fig.1 Schematic representation of equilibrium load versus buckling deflection | 22 |
| Fig.2 Schematic representation of equilibrium load versus edge displacement | 22 |
| Fig.3 Schematic representation of the effect of imperfections | 23 |
| Fig.4 Schematic representation of the effect of initial imperfections | 24 |
| Fig.5 Relative decrease of membrane stresses with increasing edge displacement beyond buckling. Longitudinal compressive strain ϵ of a long rectangular plate of width b | 25 |
| Fig.6 Shortening of a plate due to buckling | 26 |
| Fig.7 Shear angle of a rectangular plate following from oblique waves | 26 |
| Fig.8 Wave pattern for large strain ratio γ/γ_b | 27 |
| Fig.9 Stiffness of buckled panels | 27 |
| Fig.10 Explosive change of buckle pattern | 28 |
| Fig.11 Effective width | 28 |
| Fig.12 Effective width in the plastic range with 24S-T (+) and 75S-T (0) clad and unclad material | 29 |
| Fig.13 Effective width in the plastic range with 2S-1/2H clad material | 30 |
| Fig.14 Tangent to post buckling curve at buckling load for cylindrical panels | 31 |
| Fig.15 Effect of imperfections on maximum load in the initial post buckling range | 31 |
| Fig.16 Schematic representation of post-buckling behaviour in advanced stage for narrow cylindrical panels | 32 |
| Fig.17 Direct stresses of a skin plate by thermal and mechanical load | 33 |
| Fig.18 Temperature distribution in the plate $y = (T_c - T_e) \left(\frac{2z}{c} \right)^n$ | 33 |
| Fig.19 Load strain curve with thermal conditions. \bar{P} , $\bar{\epsilon}$ applies to non-thermal conditions. P , ϵ applies to thermal conditions | 33 |

NOTATION

| | |
|---------------------------|---|
| A | strain energy |
| P | load |
| R | radius of cylindrical panel |
| T | temperature |
| b | panel width |
| b_e | effective width |
| b | as a suffix denotes the quantity concerned at buckling load |
| u | edge displacement |
| α | coefficient of thermal expansion |
| γ | overall shear of a panel, following from edge displacement |
| ϵ | overall longitudinal compressive strain, edge strain |
| θ | curvature parameter defined by Equation (9) |
| σ | compressive stress |
| σ_e | edge stress |
| τ | shear stress |
| ϵ/ϵ_b | strain ratio |
| γ/γ_b | strain ratio |
| $(\alpha T)_{av}$ | average value of αT over plate width |
| $\bar{P}, \bar{\epsilon}$ | P, ϵ in isothermal condition |
| q | forces (with suffix) |

POST BUCKLING BEHAVIOUR OF STRUCTURES

A. van der Neut*

1. INTRODUCTION

Engineers have for a long time been interested in the buckling of structures, but they did not pay attention to the behaviour of structures after the onset of buckling. The accepted idea was that the buckling load was the ultimate performance of the structure, that for reasons of safety the actual load had to be kept far below what was called the critical load and that research on post-buckling behaviour had no practical importance.

The experience that some structures, such as axially compressed cylindrical shells, failed at loads far below the theoretical buckling load was attributed to inevitable imperfections in the shape. This seemed to support the appreciation of critical loads and was answered by the introduction of an ample factor of safety. On the other hand the experience that other structures, for instance, plates supported at their edges, were able to sustain loads far in excess of their buckling load, was considered with suspicion and did not upset the belief that the loads should be kept below the critical load.

Aeronautical engineers, however, could not close their eyes, either to the treacherous characteristics of shells or to promising phenomena of plates, since they were living under the ever increasing demands of weight economy in structures which were highly liable to buckling.

When applying plates as structural elements of aircraft, these plates tend to be thin and consequently buckling stresses are much smaller than the allowable stresses for more solid sections made from the same material. The most economical solution is to rearrange the distribution of material so that by having numerous supporting elements the buckling stress of the reinforced plate is well up to the material strength. With heavily loaded aircraft structures this can be achieved, as in the cases of the main structure of large aspect ratio wings and of thin wings for large wing loading and load factor. However, with plates, where the load per unit of length is relatively small, a design in which buckling would be admitted only at the ultimate load would be uneconomical.

Such conditions are present with lightly loaded wing structures and the majority of fuselage structures. Here plate buckling has been accepted, in many cases even in horizontal flight, since this type of buckling is not catastrophic and, moreover, the plate takes its share in supporting load increments due to manoeuvres and gusts. Such a plate can be regarded as remaining a structural element even after buckling. To exploit this favourable characteristic, aeronautical engineers had to investigate this 'post-buckling behaviour' of plates.

No knowledge acquired by other branches of engineering was available. Although civil engineering was confronted with the same type of prob' , for instance deep

*Professor, Technische Hogeschool Delft Vliegtuigbouwkunde, Delft, Nederland

I-beams in bridges, the need for weight economy was not so pressing there that post-buckling use was acknowledged to present a sound engineering philosophy.

It cannot be denied that some dawn of consciousness of post-buckling theory existed in naval architecture. When the steel skin of ships was reinforced by longitudinals, a small width of plate was considered to be effective as an addition to the section of the longitudinal. However, due to the Classification Rules, strength analysis could not lead the way to weight saving structures and the investigation of post-buckling behaviour was not undertaken. So, as in other cases, to aeronautics fell the honour of penetrating the bramble-bushes and awakening Sleeping Beauty with its scientific kiss. If this metaphorical language calls for a personification there is no doubt that this romantic part may be attributed to H. Wagner, who published in 1929 his wellknown work on the tension field of spar webs¹. This paper indeed is the charter legitimating the use of buckled plates as structural elements for carrying shear loads in spar webs and in the skins of wings and fuselages.

In the early *thirties* the American aircraft industry introduced a new and highly important application of post-buckling with so-called stressed-skin structures. Supporting the skin by numerous longitudinal stringers the skin, though buckled, could be made effective in carrying longitudinal compression. Knowledge of the behaviour of this type of structure started by being empirical but was soon supplemented from analytical sources. This paper gives a survey of the state of science in the field.

It must be admitted that the importance of post-buckling is past its prime of life. As previously stated, many modern aircraft structures are so heavily loaded, or need so much stiffness, that their compactness yields high buckling stresses. Smoothness of surface required for aerodynamic performance of high-speed aircraft can be another reason for the expurgation of buckled elements. However, some bright hopes for the participation of post-buckling behaviour in the supersonic age are coming from the complications to structural engineering caused by kinetic heating during transient conditions. Temporarily high compressive stresses are being generated in wing skins, which, added to the aerodynamic loads, can bring the skin into the post-buckling range. Since this situation is present only during a small percentage of the total flight time (the heating-up period), drag increase by surface waviness is possibly not prohibitive, provided this waviness does not give rise to violent vibrations. Aerodynamicists will have to supply the structural engineer with data on the amount of surface waviness which can be tolerated.

2. VARIETIES OF POST-BUCKLING BEHAVIOUR

Buckling can occur only in structural elements having at least one dimension which is much smaller than its other dimensions, as, for instance, in the case of a column, plate or shell where the thickness is small compared to the length or width. These structures are liable to buckling only when their external load is such that the internal forces act in the direction of the large dimensions, for example, a normal force in a column, or normal or shear forces in plates and shells. This type of load deforms the structure in the direction of the large dimensions and no displacement occurs in the direction of the small dimension. In other words the type of load which can bring about buckling does not cause deflections before buckling starts.

At the buckling load the state of strain is no longer unambiguously determined. Besides the pre-buckling deformation another deformation may occur, the buckling mode. The characteristic difference between this deformation and the pre-buckling deformation is that displacements in the direction of the small size occur, e.g. the structure deflects.

According to the classic theory of buckling the magnitude of this deflection is not unambiguously determined. Any deflection, provided it is of the first order of magnitude, is in equilibrium with the buckling load. This means that the tangent to the load-vs-deflection curve is horizontal (Fig. 1). Post-buckling behaviour now relates to the character of the load-deflection curve at finite buckling deflections. In Figure 1 the various possibilities of behaviour have been schematically represented. Curve a applies to columns, where the load is constant over a wide range of deflections provided no plastic deformation occurs at the point of maximum stress. Due to the deflection, the ends of the column are approaching each other. This end displacement is a quadratic in the deflection. Figure 2a represents the load-vs-shortening of the column. With increasing compression of the column the end load remains constant. As soon as plastic deformation starts the load required for equilibrium decreases, and the column collapses (dotted line in Figures 1 and 2).

This last type of behaviour has something in common with type b, which applies to shells, where with increasing deflection the load required for equilibrium decreases.

It has appeared from studies on the stability of equilibrium at buckling load that the classic theory is wrong in its statement that the structure at the buckling load is in equilibrium when the deflections are infinitely small. H.L. Cox² and von Kármán and Tsien³ discovered that some non-linear effect, overlooked so far, was of vital importance. The tangent to the load-deflection curve is not horizontal⁴; therefore as soon as deflections occur under a load equal to the buckling load P_b , the positive difference between P_b and the equilibrium load accelerates buckling, resulting in explosive collapse. Only after attaining very large deflections does the equilibrium load pass through a minimum. The highly unstable character of the structure at the load P_b means that its post-buckling behaviour is structurally useless.

The last possibility represented by curve c shows that the equilibrium load increases with increasing deflection. Since the load increment as well as the edge displacement increment are proportional to the square of the deflection, the load-vs-edge displacement curve shows a positive slope at P_b . This case applies to flat plates supported at their edges and to slightly curved panels.

So far it has been assumed that the structural members under consideration are perfect, which means that they do not deflect at loads below P_b . When the structure is initially imperfect - for example, the column not absolutely straight, the plate not absolutely plane and the shell showing analogous deviations, it deflects before the load P_b is reached. The behaviour of the three types of structures is illustrated by Figures 3 and 4. There is no longer a critical load at which buckling starts. P_b for the perfect structures a and b are replaced by a load which is close to P_b in the case of the column but much smaller than P_b in case b. Case c on the contrary shows a gradual increase of load with increasing deflection or edge displacement. The transition point between pre-buckling and post buckling behaviour vanishes. Moreover, at loads far beyond P_b the curve approaches the curve for the perfect structure.

Since imperfections cannot be completely avoided in actual structures the liability of type b to imperfections is a warning to keep actual loads far below P_b . This puts the use of structures in the buckled state still further out of consideration. However, with type c, imperfections do not alter the conclusion that these structures are useful in the post-buckled range.

Further attention will be restricted to the study of structure type c.

3. THE PHYSICAL CONCEPT OF POST BUCKLING BEHAVIOUR

Consider a perfect structure, such as a flat plate loaded up to the buckling load P_b . Then the stresses in the plate are σ_b and its edge displacements in the plane of the plate are u_b . Now let the edge displacements increase proportionally, yielding $u = \lambda u_b$. Then the plate remaining flat has the stresses $\sigma = \lambda \sigma_b$ and the required edge load is $P = \lambda P_b$.

This state of stress and strain is unstable. Allowing the plate to deflect decreases the strain energy and therefore the undeflected cannot be maintained and the plate buckles in such a pattern that the equilibrium is stable. The stability criteria for prescribed edge displacement is that the strain energy is a minimum; any state of strain adjacent to the stable one yields larger strain energy. Hence the problem when investigating post-buckling behaviour is to find for given edge displacements the buckle pattern with the least strain energy. Denoting the strain energy as A , this condition yields $\delta A = 0$ for any variation of the displacements which is continuous and complies with the kinematic boundary conditions. This variational problem yields equations which are identical with the equations of equilibrium for the deformed element of the plate.

The strain energy of the buckled plate consists for one part of 'extensional' energy - the work done by the membrane stresses over the extensions of the middle surface of the plate, - and for another part of 'inextensional' energy - the work done by the bending moments over the curvatures of the deflected plate.

In the undeflected state the strain energy is completely extensional. Therefore since the total strain energy in the buckled state is less, buckling is the way in which the plate gets rid of much of its extensional energy. This is the general trend governing the buckle pattern preferred by the plate; at the expense of some bending energy the extensional energy is kept down. The more the plate proceeds in the post-buckling range the more the ratio of extensional energy to inextensional energy decreases. The wave pattern tends towards a developable surface⁵.

The suppression of extensional energy means that the membrane stresses are kept down.

The statements are illustrated in Figure 5 by the case of a plate strip (width b) simply supported at its longitudinal edges and longitudinally compressed. With increasing overall strain ϵ the relative magnitude of the membrane stresses decreases, the centre part of the plate where the membrane stresses are small becoming wider.

This means that the buckling pattern flattens more and more with increasing strain ratio ϵ/ϵ_b . This is evident from Figure 6. Shortening of a longitudinal fibre of the plate is accomplished without excessive compression of its centre line just by following a curved course. This shortening effect is a consequence of the finiteness of the deflections and is a quadratic in the amplitude of the waves. Since the longitudinal edges of the plate are supported they cannot escape compression by deflecting and they have to carry large compressive stresses.

Due to the continuity of the deflection this edge effect extends over some width of plate. Here the membrane stresses are larger than the buckling stress. This explains why P exceeds P_b . A further conclusion is that the distribution of P over the plate width depends on the degree of buckling; therefore analysis of post-buckling behaviour, cannot start from given edge stresses but must start from given edge displacements or from a given edge load.

The favourable behaviour of plates after buckling proves to be a consequence of the fact that the edges are supported. If the longitudinal edges were free to move they would deflect as well as the centre part and the load carried after buckling would be constant, just as in the case of a column.

When the plate is distorted in shear the same general trend applies. The shear angle γ formed by the edges of a rectangular plate is composed of two parts of different origin (Fig. 7). For one part it is the shear of the middle surface of the plate due to its membrane stress. For the other part it follows from the oblique wave pattern. By folding a rectangular plate in oblique waves the length in the direction normal to the wave crest is decreased; this makes the length of the diagonals of the rectangle unequal and the rectangle changes into a parallelogram. Figure 7 does not account for edge support by which the waves are being suppressed in the edge region. Figure 8 shows how the plate manages to make the transition between its straight edges and the washboard pattern of its centre part. Whereas in Figure 7 the wave formation is such that the plate is still a developable surface this is no longer the case when the edges are kept straight. The main part of the plate is a developable surface, but the edge region is not, but near the edges the membrane stresses vary rapidly yielding a double curved surface. This is evident from the fact that plate sections parallel to the sides of the plate differ in developed length. In the case where the plate is supported by completely stiff edge members the projected length of the plate section does not change. Hence the true length of a buckled plate strip is longer than that of the edge strip, its extension being proportional to the square of the amplitude. Therefore the buckle pattern induces tensile stresses in the plate parallel to the edges, these stresses falling off towards the edges.

If the plate were to remain flat it would sustain at the given strain γ the very large membrane stress $G\gamma$, yielding large extensional energy. In changing over to a buckled shape it does not completely get rid of membrane stresses, but it brings them down to a lower level with correspondingly smaller extensional energy.

Another characteristic feature of the tendency to keep extensional energy down can be noticed, namely, the constant level of membrane stresses over a large part of the plate. Since the strain energy has something to do with the square of the stresses and $\int \sigma^2 dx > \int \sigma_{av}^2 dx$ the strain energy is being reduced when σ keeps close to the

average stress. Here again the edge support is the basic condition by which the plate can sustain loads in excess of P_b . Without edge-support the panel could be sheared according to Figure 7 by oblique waves. With vanishing bending stiffness the shear deformation could be obtained without any shear force. In the case of Figure 8 the strain ratio γ/γ_b is large and the plate has got far into the post-buckled range. When the ratio is close to unity the buckle pattern is quite different and tends to the buckle pattern at P_b . Then the surface has double curvature everywhere and the developable central part vanishes.

Returning now to general considerations some further statements can be made.

Since at given edge displacements the strain energy is smaller in the buckled state than in the flat state the external load required for this edge displacement is actually smaller than in the absence of buckling. It follows that the post-buckling stiffness, being the ratio between load and edge displacement, is smaller than the pre-buckling stiffness. This is demonstrated by Figure 9. The ratio of the increments of load and edge displacement defines the stiffness of the buckled structure with respect to incremental deformation. It is still smaller than the post buckling stiffness. This incremental stiffness is important when considering the stability of the complete structure, namely, general stability and aero-elastic stability.

When loading a plate in the post-buckling it may happen that the wave pattern changes with explosive violence. Obviously the preceding pattern contained more energy than the final one, the excess energy causing the bang. This seems to be in contrast to the statement that the way in which a structure distorts is governed by least energy. The following explanation of this phenomenon can be given.

With increasing edge displacements the buckle pattern changes continuously. It starts off with a pattern corresponding to the buckling mode and the shape of the wave modifies gradually. The tendency towards reduction of extensional energy at the expense of more bending energy involves a tendency towards reduction of wave length. The wavelength however cannot change freely due to the edge conditions imposed. For instance, with a rectangular plate of aspect ratio 4 loaded in compression the buckling mode comprises 4 half-waves. Thus number of waves cannot change continuously, the only choice being an integral number of half-waves. Therefore, while increasing the edge displacements the plate sticks to 4 half-waves, even when at a certain strain ratio 5 half-waves would yield less strain energy. The panel will jump into the pattern with lower energy level only when its equilibrium gets unstable. This will not necessarily occur at the intersection of the strain energy curves of Figure 10.

The transition point depends on the power of the disturbance which forces the plate out of its stable n -wave configuration through intermediate configurations into the stable configuration with more waves. Such disturbances are present in the form of imperfections of the plate. Dependent on the magnitude of the imperfection the jump-over occurs at a smaller or larger edge displacement. When the imperfection is very weak it may be that the transition is retarded so much that the $(n+1)$ -wave pattern is passed over and the wave number changes from n immediately into $n+2$ or more.

Summarizing:

- (a) Edge support is the basis of plate performance beyond buckling.

- (b) Analysis of post-buckling behaviour must start from given edge displacements and not from given edge stresses.
- (c) The equilibrium equations are non-linear in the deflection.
- (d) For given edge displacements the wave pattern is such that strain energy is a minimum.
- (e) The plate disposes of much of its strain energy by buckling.
- (f) Extensional energy is kept down at the expense of inextensional energy the more the plate proceeds into the post-buckling range. This reduces the membrane stresses.
- (g) The buckle pattern changes with increasing edge displacements, tending to a wide centre part with constant small membrane stress and developable surface, to narrow edge regions with large membrane stress and double-curved surface, and to a short wavelength.
- (h) With finite plate length explosive changes of wave pattern towards increasing wave number can occur.
- (i) The stiffness reduces the more the plate proceeds into the buckled range.

4. THE EQUATIONS GOVERNING POST-BUCKLING BEHAVIOUR OF FLAT PLATES

Denoting by u , v , w the displacements in the directions of the axes of x , y , z respectively, where the axis of z is normal to the plate, the equations of equilibrium of a plate element, comprising the full thickness t , are

$$q_x \equiv - \frac{Et}{1-\nu^2} \left[\frac{\partial \epsilon_x}{\partial x} + \nu \frac{\partial \epsilon_y}{\partial x} + \frac{1}{2}(1-\nu) \frac{\partial \gamma}{\partial y} - \alpha(1+\nu) \frac{\partial T}{\partial x} \right] = 0 \quad (1a)$$

$$q_y \equiv - \frac{Et}{1-\nu^2} \left[\frac{\partial \epsilon_y}{\partial y} + \nu \frac{\partial \epsilon_x}{\partial y} + \frac{1}{2}(1-\nu) \frac{\partial \gamma}{\partial x} - \alpha(1+\nu) \frac{\partial T}{\partial y} \right] = 0 \quad (1b)$$

$$q_z \equiv D \nabla^4 w + \alpha T \frac{Et}{1-\nu^2} \nabla^2 w - \frac{Et}{1-\nu^2} \left[(\epsilon_x + \nu \epsilon_y) \frac{\partial^2 w}{\partial x^2} + (\epsilon_y + \nu \epsilon_x) \frac{\partial^2 w}{\partial y^2} + (1-\nu) \gamma \frac{\partial^2 w}{\partial x \partial y} \right] = 0 \quad (1c)$$

where the strains are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2$$

$$\gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

$$D = \frac{Et^3}{12(1-\nu^2)}$$

Equations (1a) and (1b) are quadratic and Equation (1c) is cubic in w . When T is not constant, one coefficient of Equation (1c) is a function of x and y . Exact solutions of these equations are in general difficult to obtain and approximate solutions must be established.

Introducing the stress function F defined by

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} \quad \tau = - \frac{\partial^2 F}{\partial x \partial y}$$

Equations (1a) and (1b) can be replaced by the compatibility condition

$$\nabla^4 F + E\alpha \nabla^2 T - E \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] = 0 \quad (2a)$$

Then (1c) becomes

$$q_z = D \nabla^4 w - t \left(\frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right) = 0 \quad (2b)$$

These equations are known as the von Kármán-Tsien-equations^{3,6,7}. As in the case of the previous group they are non-linear and can not be solved exactly. When edge displacements u , v are given, as is usual in this type of problem, the boundary conditions for F must be derived from the following stress-strain relations:

$$\left. \begin{aligned} E \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 - \alpha T \right] &= \frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2} \\ E \left[\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 - \alpha T \right] &= \frac{\partial^2 F}{\partial x^2} - \nu \frac{\partial^2 F}{\partial y^2} \\ \frac{E}{2(1-\nu)} \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] &= - \frac{\partial^2 F}{\partial x \partial y} \end{aligned} \right\} \quad (2c)$$

Since approximate solutions can be obtained on the basis of the principle of minimum strain energy the expression for strain energy per unit area will be given. It is⁸

$$A = \frac{Et}{2(1-\nu^2)} \left\{ (\epsilon_x + \epsilon_y)^2 - 2(1-\nu)(\epsilon_x\epsilon_y - \frac{1}{4}\gamma^2) - 2(1+\nu)\alpha T(\epsilon_x + \epsilon_y) + \right. \\ \left. + 2(1+\nu)\alpha^2 T^2 + \frac{t^2}{12} \left[(\nabla^2 w)^2 - 2(1-\nu) \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) \right] \right\} \quad (3)$$

5. METHODS FOR SOLUTION

Approximate solutions always start with some assumption about the function representing the deflection w . Apart from mathematical reasons it is the most spectacular component of the displacement and therefore the least difficult one to estimate. The assumption of w contains at least one and preferably more undetermined parameters p .

One method of solution is along the following lines, using the Equations (2). Substituting w into (2a) this equation is a linear equation in F and can be solved exactly or if necessarily approximately, taking due account of the boundary conditions. Then F is known as a function of the parameters. Thereupon w and F are substituted in (2b). The solution being an approximate one the lefthand side of (2b) cannot be made to vanish by choosing a finite number of parameters p and some method must be devised for determining the parameters such that (2b) is satisfied in the average.

For this the Ritz-Galerkin method is preferable. It yields the equations

$$\iint q_z \frac{\partial w}{\partial p} dx dy = 0 \quad (4)$$

the number of which is equal to the number of parameters. If F is the exact solution of (2a) this method is equivalent to the Raleigh-Ritz method. It yields those values of the parameters p for which the strain energy is less than with any other choice of parameters. Since the solution is an approximate one the energy is larger than the actual amount when the plate would be free to find the wave pattern corresponding to minimum energy. Therefore the stiffness obtained in this way is larger than the actual stiffness.

It is a pity that exact solutions for F can be obtained only for very simple functions of w , and that even then their computation is tedious⁸.

When F is an approximate solution of (2a) nothing can be concluded about the nature of the approximation. Since the compatibility condition is not satisfied the strains are not compatible, whereas the Raleigh-Ritz method applies to compatible strains only. Approximating F by using the minimum complementary energy theorem, the approximation obtained for the extensional energy is too small. This is only one part of the energy; the other part is being overestimated by using Equation (4).

So with the mixture of the theorems of least complementary energy and least strain energy in an unrigorous way no definite conclusion is possible on the nature of the approximation.

In this respect a method which is based on one theorem only is preferable, the theorem of minimal strain energy. This solution starts from Equations (1) and applies the Ritz-Galerkin method, or it starts from Equation (3) and applies the Raleigh-Ritz method. These methods are mechanically identical. Besides an assumption for w , assumptions have to be made for u and v . These assumptions include some parameters p . Then the strains can be computed and consequently from Equation (1) the 'additional forces' q_x , q_y , q_z or from Equation (3), the strain energy.

Using the Raleigh-Ritz method, the equations from which the parameters p can be solved are

$$\iint \frac{\partial A}{\partial p} dx dy = 0 \quad (5)$$

Using the Ritz-Galerkin method, the equations for the determinations of the parameters p are

$$\iint \left(q_x \frac{\partial u}{\partial p} + q_y \frac{\partial v}{\partial p} + q_z \frac{\partial w}{\partial p} \right) dx dy = 0 \quad (6)$$

q_x , q_y , q_z are the additional forces per unit area required for equilibrium at the approximate displacements u , v , w . The condition expresses that the potential energy of this additional load system does not change when the parameters vary. It can be shown that this condition is equivalent to the minimum potential energy theorem, which forms the basis of Equation (5).

When u , v , w do not satisfy the dynamical boundary conditions, equilibrium requires additional edge forces. In that case the left hand side of Equation (6) should be complemented by the contour integral of the product of this additional edge force and the derivative to p of the corresponding edge displacement.

The difficulty with these methods is that an appropriate guess of u and v has to be made. Of course the adaptability of the guess can be improved by increasing the number of parameters, but this soon leads to a prohibitive number of non-linear simultaneous algebraic Equations of the form (5) or (6). For this practical reason u and v must be constructed by means of the parameters adopted for w . The best way to do this is to make assumptions on the distribution of the strains ϵ_x , ϵ_y , γ , from which u and v can be derived, taking into account the given edge displacements.

This procedure comes close to the procedure which is applied when (2a) is solved approximately. Then appropriate stress functions F have to be chosen and this is essentially a guess of stress distribution or strain distribution. Then however the possibility exists of combining a certain number of these stress functions, which gives the approximation more adaptability, provided the coefficients of the stress functions become simple functions of the parameters p . Unfortunately this latter condition is usually not satisfied, which means again that the evaluation of the equations is getting prohibitively laborious.

6. AVAILABLE SOLUTION FOR BEHAVIOUR AFTER BUCKLING

6.1 Longitudinal Compression (Effective Width)

6.1.1 Flat plates

A longitudinally compressed plate carries the load

$$P = t \int_0^b \sigma dy$$

When the edge strain (compressive) is ϵ the stress in the edge stiffener is $\sigma_e = E\epsilon$. Now the 'effective width' b_e is defined by

$$P = \sigma_e t b_e$$

Hence

$$\frac{b_e}{b} = \int_0^1 \frac{\sigma}{\sigma_e} d(\gamma/b)$$

can be called the post-buckling effectiveness of the plate.

Von Kármán answered the urgent need for knowledge on effective width by devising an engineering guess based on very much simplified considerations. Although this was an ingenious close guess, giving (Fig. 11)

$$\frac{b_e}{b} = \left(\frac{\sigma_b}{\sigma_e} \right)^{1/2} \quad (7)$$

its reliability is not self-evident.

The first thorough investigation is due to Marguerre and Trefftz^{8,9}. They assumed the wave pattern

$$w = f \sin \frac{\pi y}{b} \sin \frac{\pi x}{L}$$

where f and L are the parameters. This wave form comprises the buckling mode, so it will yield a good approximation for small strain ratios ϵ/ϵ_b . However, for $\epsilon/\epsilon_b = \infty$ it yields $b_e/b = 1/3$, whereas for physical reasons b_e/b should vanish. This discrepancy is a consequence of expressing the wave form as a function of y , which does not permit flattening at larger strain ratios. H.L. Cox¹⁰ accounted for this effect, assuming over the centre part a constant wave depth and sinusoidal deflection in the edge region. His parameters are the wave amplitude and the width of the edge region, the wave length being constant. His application of Raleigh's method is not rigorous, since part of the extensional energy is neglected. In this respect the energy is being underestimated; however, the approximation of the wave form has the opposite effect.

The most advanced investigation has been done by W.T. Koiter¹¹. Unfortunately, this work has only been published in Dutch, so that it is little known. Koiter followed Cox's suggestion for the wave pattern, allowing the wave length, however, in addition to the wave depth and the width of the edge region, to be a parameter. Also, he did not neglect part of the energy.

Several assumptions were made for the displacement functions: for the simply supported plate (4), one of them having continuous $\partial^2 w / \partial y^2$ at the transition of the centre part and edge region; for the clamped plate (2); and for the elastically restrained plate (3).

The good agreement between the results obtained with different wave forms indicates that the approximation is satisfactory and that the results are not sensitive to slight alterations in the wave form assumed. Moreover, the effective width as a function of the strain ratio proved to be practically independent of the boundary conditions. Hence the effective width for simply supported, elastically restrained, or clamped edges can be given by the same function of ϵ/ϵ_b , viz.

$$\frac{b_e}{b} = 1.20 \left(\frac{\epsilon_b}{\epsilon} \right)^{2/5} - 0.65 \left(\frac{\epsilon_b}{\epsilon} \right)^{4/5} + 0.45 \left(\frac{\epsilon_b}{\epsilon} \right)^{6/5} \quad (8)$$

where ϵ_b is the buckling strain pertaining to the actual boundary conditions. Equation (8) gives a representation of this function with no more than 1% error. Figure 11 shows that the agreement between experimental results for various edge restraints and Koiter's analytical results is quite satisfactory. Since the approximation must yield a too large effective width it is somewhat surprising to find that most of the experiments fall above the theoretical curve. The most acceptable explanation will be that in actual panels the wave length reduction with increasing strain ratio will be retarded, whereas Koiter's computations apply to the infinitely long plate, where nothing prevents a gradual change of wave length.

Koiter's theory has the advantage that it holds for any strain ratio between 1 and ∞ . More accurate investigations have been made by Hemp¹², applying to strain ratios a little over 1. They are not discussed here, because the inevitable imperfections of plates affect the behaviour of plates in this region sensibly (compare Figure 4), reducing the practical importance of more accurate knowledge.

During the last few years the Netherlands Aeronautical Laboratory has proceeded with its investigations on effective width, extending it to the behaviour of plates in the plastic range by experiments on 24S-T and 75S-T, clad and unclad^{13, 14}. The range of strain ratios was $1 < \epsilon/\epsilon_b < 50$ and ϵ_p/ϵ_b was in the range $1.5 < \epsilon_p/\epsilon_b < 20$, where ϵ_p is the strain at which the elastic limit of the material is exceeded. Referring the effective width to the edge stress $\sigma_e = E_s \epsilon$, the results suggest the empirical conclusion that Koiter's Equation (8) may be extrapolated into the plastic range. Some representative results are shown in Figure 12.

Some series of tests were also carried out on 2S-1/2 H material so as to investigate post-buckling behaviour with materials having a more gradual transition between the elastic and plastic parts of the stress-strain curve¹⁴. In this case the effective width in the plastic range is greater than that given by the theoretical curve for elastic materials. Some results are shown in Figure 13.

6.1.2 Narrow cylindrical panels

Quite recently Koiter investigated the initial post-buckling behaviour of narrow cylindrical panels such as occur in wings and fuselages with longitudinal stiffeners¹⁵. All constraining effects of the stiffeners other than the complete suppression of radial deflection were neglected.

The tangent to the post-buckling curve at buckling load has been established. It is given by

$$Ebt\epsilon = P + \frac{P - P_b}{\phi(\theta)} \quad (9)$$

where
$$\theta = \frac{1}{2\pi} [12(1 - \nu^2)]^{1/4} b(Rt)^{-1/2}$$

$$\phi(\theta) = 1 - 2\theta^2 - 8 \sum_{n=1}^{\infty} [(n^2 + 1)^{-2} + (n^2 + 1)^2 \theta^{-4} - 1 - \theta^4]^{-1/2}$$

$$P_b = Ebt(1 + \theta^4)\epsilon_b \quad \text{for } \theta \leq 1$$

and is shown in Figure 14.

In wing panels the values of θ are usually considerably smaller than 0.5 and as appears from Figure 14 the post-buckling behaviour is very similar to that for flat plates, where $\theta = 0$.

In fuselage panels the values of θ range from approximately 0.4 to about 2, so that usually the initial post-buckling behaviour differs considerably from that of a flat plate.

For $\theta < 0.3$ $dP/d\epsilon$ is practically equal to that for $\theta = 0$; curvature does not affect the post-buckled behaviour. For $\theta > 0.5$ $dP/d\epsilon$ starts to decrease rapidly with increasing θ , passing at $\theta = 0.65$ through zero and becoming negative.

The horizontal tangent represents the limiting case of a stable post-buckled behaviour for a prescribed load. At $\theta = 0.77$ the tangent is vertical and at $\theta = 1$ the behaviour is very close to that of a full cylinder.

These figures apply to the panel the edges of which are simply supported. It may be expected that initial post-buckling behaviour with clamped or elastically restraint edges will be stable up to higher values of θ , but the general trend will be the same.

The effect of initial sinusoidal imperfections at small deflections has been investigated. Then the load up to which the behaviour is stable is lowered

considerably when $\theta > 0.65$ (compare Figure 4b). This 'buckling load' depends on the ratio of the amplitude a of the initial waves to plate thickness. Figure 15 shows this for two values of θ .

As for the more advanced post-buckling stages it seems reasonable to expect that it approaches that of a flat plate of equal width.

Considering a free panel simply supported at its edges, the initially curved plate tends in its centre part to become a developable surface, as with the initially flat plate. Supposing that the curved plate when buckled has the same shape as the initially flat plate, the strain energy per unit length in the first case is $\frac{1}{2} Ebt \theta^4 \epsilon_b^2$ (ϵ_b is the buckling strain of the flat plate) greater than that of the flat plate, where it is $\frac{1}{2} Ebt \epsilon_b^2 \left[\frac{1}{3} + \frac{2}{3} (\epsilon/\epsilon_b)^{3/2} \right]$ in the initial post-buckling stage and in the advanced stage even more than that. Hence when $\epsilon/\epsilon_b \gg 1$ the strain energies in these two cases are approaching one another, yielding equal effective widths.

With a shell running continuously over regularly spaced stringers the individual panels are not free to assume the wave pattern of the flat plate, since this would imply a discontinuity of the slopes at the stringer, the discontinuity being $b/R = 2\pi\theta^2\epsilon_b^{1/2}$. The condition of continuity means that the adjacent panels are restraining each other, the effect of which is an increase of stiffness.

Starting from the buckle pattern of the single panel, the edges have to be rotated through the constant angle $\pi\theta^2\epsilon_b^{1/2}$. This angle is of the same order of magnitude as the slope of the wave pattern with the free panel, when ϵ/ϵ_b is small. However, with a large strain ratio ϵ/ϵ_b the slope with the free panel is about $1.7\epsilon^{1/2}$; then the correction required for continuity is relatively small and so is the additional energy originating from this edge restraint.

The conclusion is that the load sustained by a curved panel is for the same edge strain somewhat greater than the load carried by a flat panel of the same width and tends to become equal at large strain ratios.

Figure 16 shows schematically the load-strain curve for plates which are unstable in the initial post-buckling range. Since imperfections cut off the peak of the load-strain curve, near the buckling load the curve for the actual plate will be still closer to that for the flat plate. So for practical purposes it is justified to identify post-buckling behaviour of narrow ($\theta \leq 1$) cylindrical panels and flat plates in spite of the initially unstable post-buckling behaviour when $\theta > 0.65$.

6.2 Flat Plates Under Shear Load

As stated before, the first analytical investigation on post-buckling behaviour of plates loaded in shear was made by H. Wagner¹. Its object is the 'complete tension field', where the plate is supposed to be unable to carry compressive membrane stresses. This means that $\gamma_b = 0$, or more precisely that the strain ratio (to which this investigation applies) $\gamma/\gamma_b = \infty$. So it represents the limiting case.

Wagner's work has been of much importance in the design of spar webs, etc., but it yields conservative design since the performance of the plate is underestimated. The

analytical investigation of the 'incomplete tension field', where the plate can sustain compressive as well as tensile stresses, starts with a paper by Kromm and Marguerre⁹ applying to the simply supported plate strip.

They assumed the wave form

$$w = f \sin \pi \frac{y}{b} \left[\sin \frac{\pi(x - my)}{L} \right] \quad (10)$$

where f , L and m are the parameters.

This wave pattern is an approximation of the huckle mode. Its nodal lines are straight, whereas the tangent of the exact nodal line is perpendicular to the edge. The buckling strain γ_b obtained from this approximate wave form is 5.7% too large.

The data obtained for post-buckling behaviour are good at strain ratios not too much greater than 1. At large strain ratios the wave pattern flattens in the centre part, which phenomenon is not represented in the pattern (10). So it is evident that Reference 9 does not yield Wagner's results in the limiting case of infinite strain ratio. This first attempt was improved by Koiter¹⁶, assuming, as in the case of the effective width problem, constant wave amplitude in the centre part (developable surface) and double curvature in the edge regions. This adds a fourth parameter to those of Equation (10), namely the width of the centre part.

Several assumptions were made for the wave pattern in the edge regions: for the simply supported plate (4), one of them having continuous curvature at the transition of centre part and edge region, another one with nodal lines ending perpendicularly to the edges; and for the clamped plate (2).

This theory yields for $\gamma/\gamma_b = \infty$ results which are in agreement with Wagner's complete tension field theory. The differences between the results obtained from the various wave patterns were not very important. One of them applying to the simply supported plate was chosen for evaluation of Koiter's equations, which has been established in References 17 and 18.

Reference 17 contains diagrams from which the relation between the overall strain ratios ϵ_1/ϵ_b , ϵ_2/ϵ_b , γ/ϵ_b and the average stress ratios $\sigma_1/E\epsilon_b$, $\sigma_2/E\epsilon_b$, $\tau/E\epsilon_b$ can be read. The indices 1 and 2 apply to the longitudinal and the lateral direction respectively. Each diagram applies to one of the $\tau/E\epsilon_b$ ratios 0.5, 10, ..., 45, 50; the coordinates are ϵ_1/ϵ_b , ϵ_2/ϵ_b ; the diagrams give curves of constant $\sigma_1/E\epsilon_b$, $\sigma_2/E\epsilon_b$ and $\tau/G\gamma$. Thus at given ϵ_1/ϵ_b , ϵ_2/ϵ_b the corresponding stress ratios $\sigma_1/E\epsilon_b$, $\sigma_2/E\epsilon_b$ and the corresponding shear rigidity reduction $\tau/G\gamma$ are found by interpolation

The field of application of these data is not limited to webs loaded in shear. Since ϵ_1/ϵ_b and ϵ_2/ϵ_b are arbitrary they apply as well to the case of stiffened plates loaded in shear and in compression or tension longitudinally and laterally.

They make it possible to establish for simply supported plates:

- (a) From given edge displacements the edge loads

- (b) From given edge loads the edge displacements
- (c) From a mixture of three given edge displacements or loads the other displacements and loads
- (d) The stiffness of the plate
- (e) The stiffness with respect to load increments, for instance the change of shear stiffness resulting from a change of longitudinal compression.

These data are strictly valid only when the stresses are below the proportional limit. The stresses can be derived from the wave form. This has been evaluated in Reference 18, which establishes the greatest effective strain ϵ_e according to the Huber-von Mises-Hencky criterion, again in diagrams applying to constant $\tau/E\epsilon_b$, where the coordinates are ϵ_1/ϵ_b , ϵ_2/ϵ_b and curves of constant ϵ_e/ϵ_b are given.

The former data apply to simply supported edges. The evaluation of plates with clamped edges has not been done systematically. It has been found that, contrary to the effective width problem, post-buckling behaviour of plates loaded in shear is not merely dependent on γ/γ_b but on the boundary conditions as well. For the same strain ratios γ/γ_b the clamped plate is closer to the complete tension field than is the simply supported plate. This difference with the effective width problem can be physically understood as follows.

At large strain ratios the strain energy of the compressed plate is mainly confined to the edge regions (large membrane stresses, large curvatures), whereas in the tension field the centre part as well contains much strain energy since the normal and shear stresses in the centre part and near the edges are equally important.

The wave form flattens at some distance from the edge, which means that the effect of boundary conditions has damped out when getting at the centre part. Therefore the strain energy of the centre part is almost independent of the boundary conditions. On the contrary the buckling strain depends on the boundary conditions. Hence the total strain energy cannot be merely a function of the strain ratio. At very large strain ratios the edge regions are narrow, so that the total strain energy is almost equal to the energy of the centre part. Then the strain energy would be almost independent of the boundary conditions. This means that for large strains the post-buckling behaviour of clamped plates is almost the same as that of the simply supported plate at the same strain.

An experimental check of theory was not possible with the tests by Lahde and Wagner¹⁹, since they investigated the plate with clamped edges. Tests on spar webs are usually confined to shear load and slight additional bending, and usually the support of the plate by stiffeners provides some restraint. So the Netherland Aeronautical Institute undertook a test program on plates, loaded in compression and shear, without restraint at the supporting members. The results have been communicated^{20, 21}. The result showed considerable scatter, more in particular at the smaller strain ratios. Obviously the imperfections of the plate have much effect on post-buckling behaviour and this scatter prevented an accurate comparison of theory and tests. However, the tests provide sufficient evidence that Koiter's theory gives an adequate picture of post-buckling behaviour over the whole range of strain ratios

and combinations of normal and shear loads, more in particular for panels having large aspect ratio. In view of scatter in post-buckling behaviour of actual plates there seems little point in further refinement of analysis. So far Koiter's theory is the only one with this wide range. Several authors made valuable contributions, like Denke^{22, 23}, Levy^{24, 25} and Bergmann²⁶, applying to the square plate or the rectangular plate with length to width ratio not larger than 2. However, the validity of these theories is restricted to small and moderate strain ratios and to panels loaded in shear only.

The practical importance of a satisfactory theory after evaluation is that it provides knowledge on stiffness of buckled plates. From this knowledge the stress-strain relations of a complete structure, composed of plates and other elements like stringers and frames, can be determined. Apart from that a satisfactory theory enables us to establish the edge displacements at which permanent wrinkling starts. However, theory is unable to predict the edge displacements at which failure occurs. When the structure fails in rivet joints before permanent wrinkling starts, the failure could be predicted by analysis. But usually rivet joints will be made stronger and their failure will be preceded by plastic deformation in the plate. In view of the complexity of the problem there are no hopes for analytical investigations on post-buckling behaviour in the plastic range. Another possible cause of failure is stiffener failure due to the combined action of normal load and flexure and torsion imposed by plate wrinkles. This phenomenon too is beyond the possibilities of analysis. Therefore knowledge of ultimate load of stiffened plates in the post-buckled range must be acquired empirically. In this respect much has been done at N.A.C.A. by Kuhn, Peterson, Levin and other. Kuhn has summarized this work in his recently published text book²⁷. These data enable us to predict ultimate loads with no more than 10% error.

6.3 Flat Plates Under Thermal Stresses

In supersonic flight the temperature of the structure rises due to kinetic heating. When flight has been maintained at constant speed during some length of time the temperature differences in the structure are small and so are the thermal stresses. However in transient conditions heat flow produces temperature differences and consequently thermal stresses, which affect both buckling and post-buckling behaviour. The skin is heated by the air, but at the plate edges part of the heat input is transferred to the interior structure, such as webs of multi-web wings. Therefore the edges have a lower temperature than the centre part. Since the longitudinal strain of the plate strip is constant throughout the width of the plate and the resultant force should vanish, compressive stresses occur in the centre part and tensile stresses occur near the edges. To this stress distribution is added the stress due to the mechanical loads on the aircraft.

Since we are interested in post-buckling behaviour, the part of the skin which should be examined is the side which is loaded in compression. Here the skin plate presents before buckling a stress distribution of the type represented in Figure 17. It has been shown²⁸ that a good approximation of the buckling mode is given by the buckling mode for uniformly stressed plates. This is a guide for post-buckling investigations. So the buckling pattern appropriate to non-thermal conditions, assumed in Reference 11, can be applied to thermal conditions.

In Reference 29 post-buckling behaviour has been investigated for plates loaded in compression and by the following temperature distribution:

$$T = T_0 + T \sin \pi y/b \quad (11)$$

or by a temperature distribution in which T is constant over part of the plate width in the centre and falls off towards the edges according to a power n (Fig. 18). Conclusions from this investigation can be given in a very simple form, since the load-strain curve appears to have the same shape as that for the non-thermal condition. Denoting by \bar{P} and $\bar{\epsilon}$ the load and compressive edge strain for the non-thermal condition, the relation between P and ϵ in the thermal condition follows from:

$$\left. \begin{aligned} \frac{P}{Ebt} &= \frac{\bar{P}}{Ebt} + \frac{1}{b} \int_0^b \alpha T \cos \frac{2\pi y}{b} dy \\ \epsilon &= \bar{\epsilon} - \frac{1}{b} \int_0^b \alpha T \left(1 - \cos \frac{2\pi y}{b} \right) dy \end{aligned} \right\} \quad (12)$$

The edge stress is $\sigma_e = E(\epsilon + \alpha T_e)$.

So the P - ϵ curve is as shown in Figure 19; it follows from the \bar{P} - $\bar{\epsilon}$ curve simply by sliding the axes over a distance, which depends on the temperature distribution. The buckling load decreased by thermal stresses. However the load increment in the post-buckled range is independent of the temperature, it depends only on the increment of edge strain beyond the critical strain. This statement is incorrect in so far it does not consider the effect of temperature on the modulus of elasticity. It must be complemented by the instruction that the \bar{P} - $\bar{\epsilon}$ curve refers to the modulus of elasticity for the temperature level under consideration.

The integral over T in Equation (12) shows, that the buckling mode across the width is proportional to $\sin \pi y/b$. When allowing for flattening of the centre part, as applied in Reference 11, the P - ϵ curve is only slightly different for $\bar{\epsilon}/\epsilon_b > 4$. With the temperature distribution (11) the temperature effect is rather pronounced; nevertheless the difference between P computed from (12) and P computed from the flattened wave shape was well within 1%. Equation (12), therefore, can be considered to hold for the whole range of strain ratios.

The simplicity of the conclusion that the relation between load and strain increment is almost unaffected by thermal conditions suggests that some physical evidence must exist. The following explanation can be given.

Consider two equal plates 1 and 2 differing only in temperature distribution - the temperature of plate 1 being constant, plate 2 having varying temperature - longitudinally compressed up to their respective buckling loads.

At the buckling load P_b the stresses are σ_b . Since the strain energy is a minimum for any variation $\delta\gamma$ of the strain γ , which is accompanied by the longitudinal edge displacement $L\delta\epsilon$,

$$\int \Sigma(\sigma_b \delta\gamma) dv = P_b L \delta\epsilon \quad (13)$$

for both plates 1 and 2, where σ_b and P_b are different. Now consider the plates both to be compressed longitudinally the same amount $\Delta\epsilon$ beyond their compressive strain ϵ_b and assume that the wave pattern of plate 2 is the same as that of plate 1, e.g. the assumption is that the wave pattern is affected negligibly by the difference of thermal conditions. Then the stress increments $\Delta\sigma$ following from $\Delta\epsilon$ are equal for plates 1 and 2. The load increment is ΔP . Since plate 1 is in equilibrium in its deflected shape

$$\int \Sigma[(\sigma_{b1} + \Delta\sigma_1) \delta\gamma] dv = (P_{b1} + \Delta P_1) L \delta\epsilon \quad (14a)$$

for any compatible system of strain variations $\delta\gamma$. The approximation for the load upon plate 2 is found from the condition that the work done by P is equal to the increase of strain energy when ϵ increases with $\delta\epsilon$. The strain variations between the states of compression ϵ and $\epsilon + \delta\epsilon$ are $\delta\gamma$. Hence

$$\int \Sigma[(\sigma_{b2} + \Delta\sigma_2) \delta\gamma] dv = (P_{b2} + \Delta P_2) L \delta\epsilon \quad (14b)$$

Substituting (13) into (14a) and (14b), and remembering that $\Delta\sigma_1 = \Delta\sigma_2$ it follows that $\Delta P_1 = \Delta P_2$ for equal compressive strain increment $\Delta\epsilon$ beyond buckling strain. This conclusion has been based upon the assumption that the wave pattern of the non-thermal case is a good approximation for the wave pattern in the thermal case. This restricts the validity of the statement that $\Delta P/\Delta\epsilon$ is unaffected by thermal conditions, but the impression is that these restrictions are not important.

Our reasoning was given with respect to the longitudinally compressed plate. However, any other edge displacement could have been taken. Hence we can extend the validity of the statement to any case of thermal load and mechanical load in shear and compression, provided the wave pattern is affected only slightly by thermal stresses.

This widens the statement to: the relation between load increment $P - P_b$ and overall strain increment $\epsilon - \epsilon_b$ is not influenced by thermal stresses. When the post-buckling behaviour is known for non-thermal conditions, the problem of post-buckling for thermal conditions is solved as soon as the buckling load P_b has been established.

For the time being the only available literature on post-buckling behaviour of thermally loaded plates is contained in Reference 7. The case considered there was devised so that experimental check of the analysis was possible.

Since the temperature distribution is far from actual conditions with kinetic heating of aircraft the results need not be discussed in detail. It suffices to say that the theoretical and the measured wave patterns agreed satisfactorily. Furthermore, Reference 7 establishes an interesting method for computing the correction of the deflections following from initial imperfections, which are of the same form as the wave pattern.

REFERENCES

1. Wagner, H. *Spars with very thin webs.* Zeitschr. für Flugtechnik und Motorluftschiffahrt 1929, pp. 200, 227, 256, 279, 306 (in German).
2. Cox, H.L. *Stress analysis of thin metal construction.* J. Roy. Aer. Soc., vol.44, 1940, p. 231.
3. von Kármán, Th.
Tsien, Hsue-Shen *The buckling of spherical shells by external pressure.* J. Aer. Sci., vol.7, 1939, p. 43.
4. Koiter, W.T. *On the stability of elastic equilibrium.* Thesis Technische Hogeschool, Delft, 1945 (in Dutch).
5. Kirste, M.L. *The developable deformations with thin plates.* Technique et Science aéronautiques, 1955, p. 192 (in French).
6. von Kármán, Th.
Biot, M.A. *Mathematical methods in engineering.* McGraw-Hill Book Co. Inc., 1940, p. 156.
7. Gossard, M.L.
et alii *Thermal buckling of plates.* NACA TN 2771, 1952.
8. Marguerre, K.
Trefftz, E. *On the performance of a plate strip beyond buckling.* Zeitschr. für angewandte Math. und Mech., vol. 17, 1937, p. 85 (in German).
9. Kromm, A.
Marguerre, K. *Post buckling behaviour of a plate strip loaded in shear and compression.* Luftfahrtforschung 1937, p.627 (In German).
10. Cox, H.L. *Buckling of thin plates in compression.* Aer. Res. Comm. R. and M 1554, 1933.
11. Koiter, W.T. *The effective width of flat plates for various longitudinal edge conditions at loads far beyond buckling load.* Nat. Luchtvaart Lab. (NLL) Report S 287, 1943 (in Dutch).
12. Hemp, W.S. *Theory of flat plates buckled in compression.* ARC R&M 2178.
13. Botman, N.
Besseling, J.F. *The effective width in the plastic range of flat plates under compression.* NLL Report 'S 445, 1954.
14. Botman, M. *The effective width in the plastic range of flat plates, under compression, (part III).* NLL Report S465, 1955.

15. Koiter, W.T. *Buckling and post-buckling behaviour of a cylindrical panel under axial compression.* NLL Report S 476, 1956.
16. Koiter, W.T. *The tension field in flat plates at loads far beyond the buckling load.* NLL Report S 295, 1944 (in Dutch).
17. Floor, W.K.G.
Burgerhout, T.J. *Evaluation of the theory on the post-buckling behaviour of stiffened flat rectangular plates subjected to shear and normal loads.* NLL Report S 370, 1951.
18. Floor, W.K.G. *Investigation of the post-buckling effective strain distribution in stiffened flat rectangular plates subjected to shear and normal loads.* NLL Report S 427, 1953.
19. Lahde, R.
Wagner, H. *Experimental investigation of tension field.* Luftfahrtforschung, vol. 13, 1936, p. 262.
20. van der Neut, A.
Floor, W.K.G. *Experimental investigation of the post buckling behaviour of flat plates loaded in shear and compression.* NLL Report S 341, 1949. Abbreviated version in Proc. of the 7th Int. Congress of Appl. Mech. London, 1948.
21. Floor, W.K.G. *Evaluation of the theory on the post-buckling behaviour of stiffened flat rectangular plates subjected to shear and normal loads, part III.* NLL Report S 367, 1950.
22. Denke, P.H. *Strain energy analysis of incomplete tension field web-stiffener combinations.* J. of the Aero. Sc., vol. 11, 1944, p. 25.
23. Denke, P.H. *Analysis and design of stiffened shear webs.* J. of the Aer. Sc., vol. 17, 1950, p. 217.
24. Levy, S.
et alii *Analysis of square shear web above buckling load.* NACA TN 962, 1945.
25. Levy, S.
et alii *Analysis of deep rectangular shear web above buckling load.* NACA TN 1009, 1946.
26. Bergman, S.G.A. *Behavior of buckled rectangular plates under the action of shearing forces.* Thesis Kungl. Tekniska Högskolan, Stockholm, 1948
27. Kuhn, P. *Stresses in aircraft and shell structures.* McGraw-Hill Book Co. Inc., 1956.
28. van der Neut, A. *Buckling caused by thermal stresses.* VTH Rep. 68 (1956). AGARD-ograph on: High temperature effects on aircraft structures, chap.9 (not yet published).
29. van der Neut, A. *Post buckling performance of plates under thermal load.* VTH Rep. 70, 1956, (not yet published).

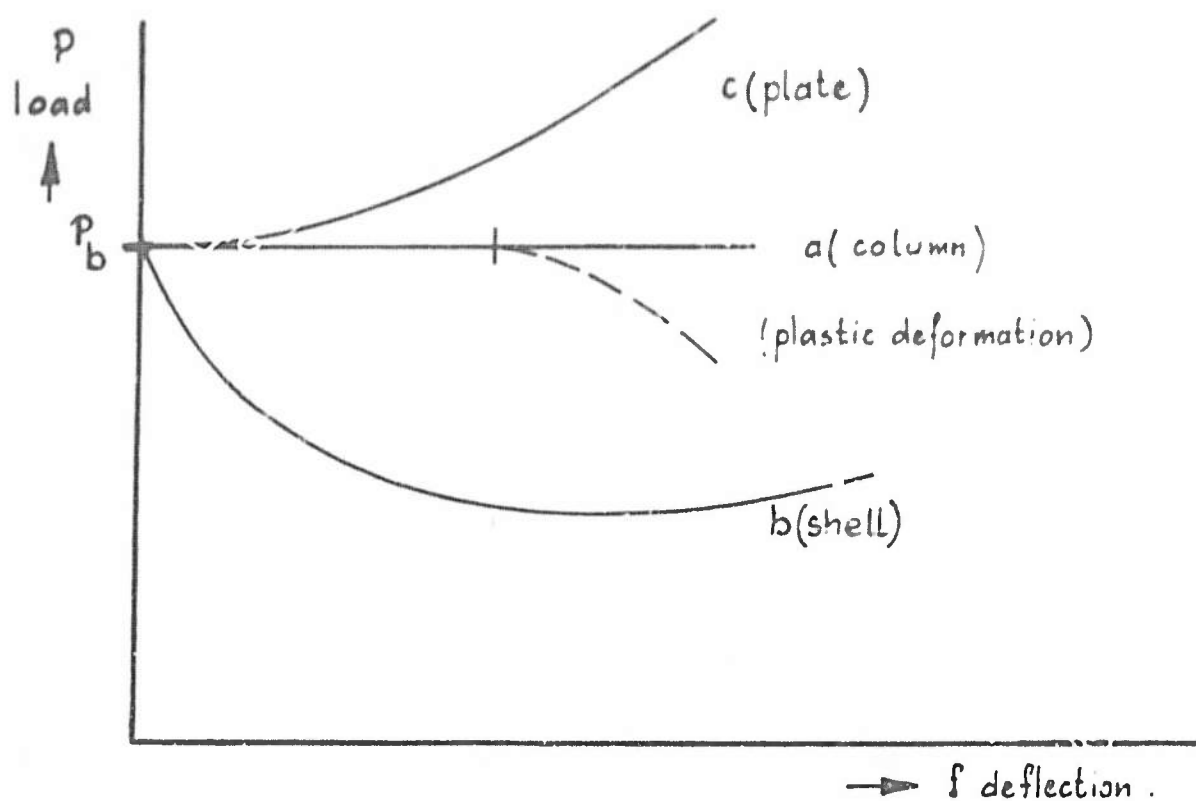


Fig. 1 Schematic representation of equilibrium load versus buckling deflection

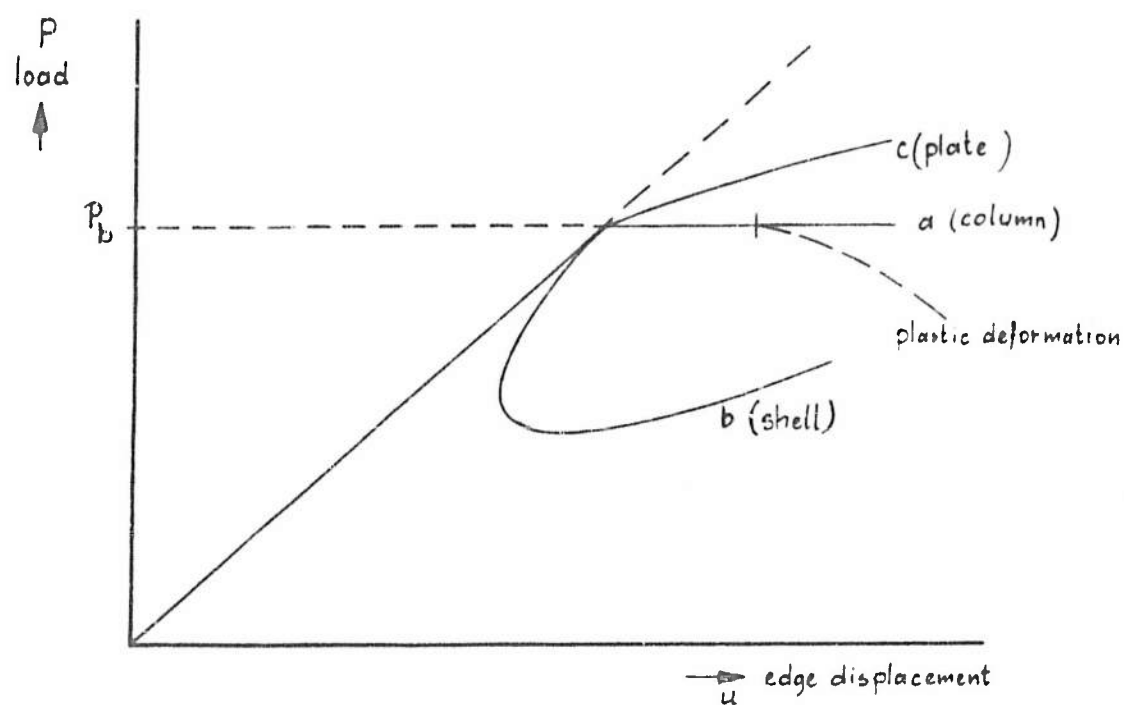


Fig. 2 Schematic representation of equilibrium load versus edge displacement

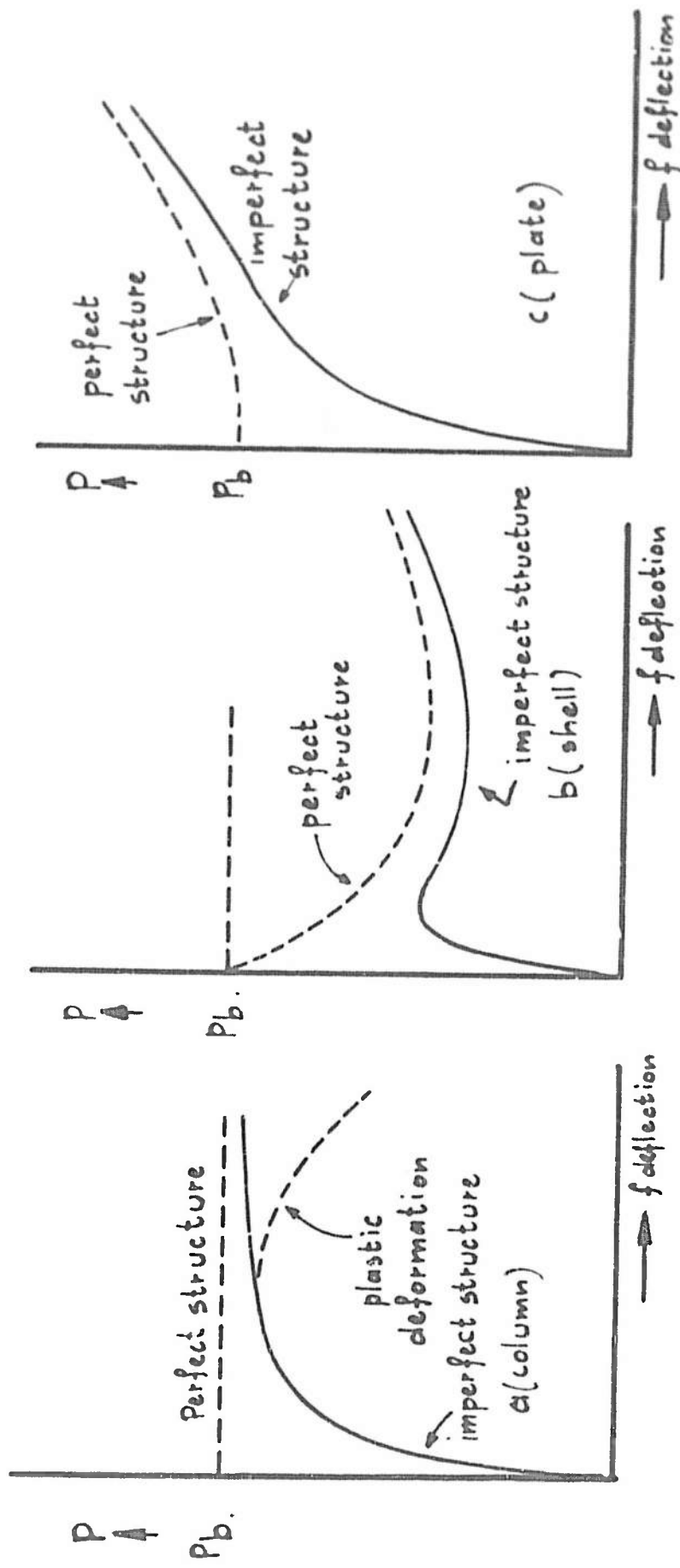


Fig. 3 Schematic representation of the effect of imperfections

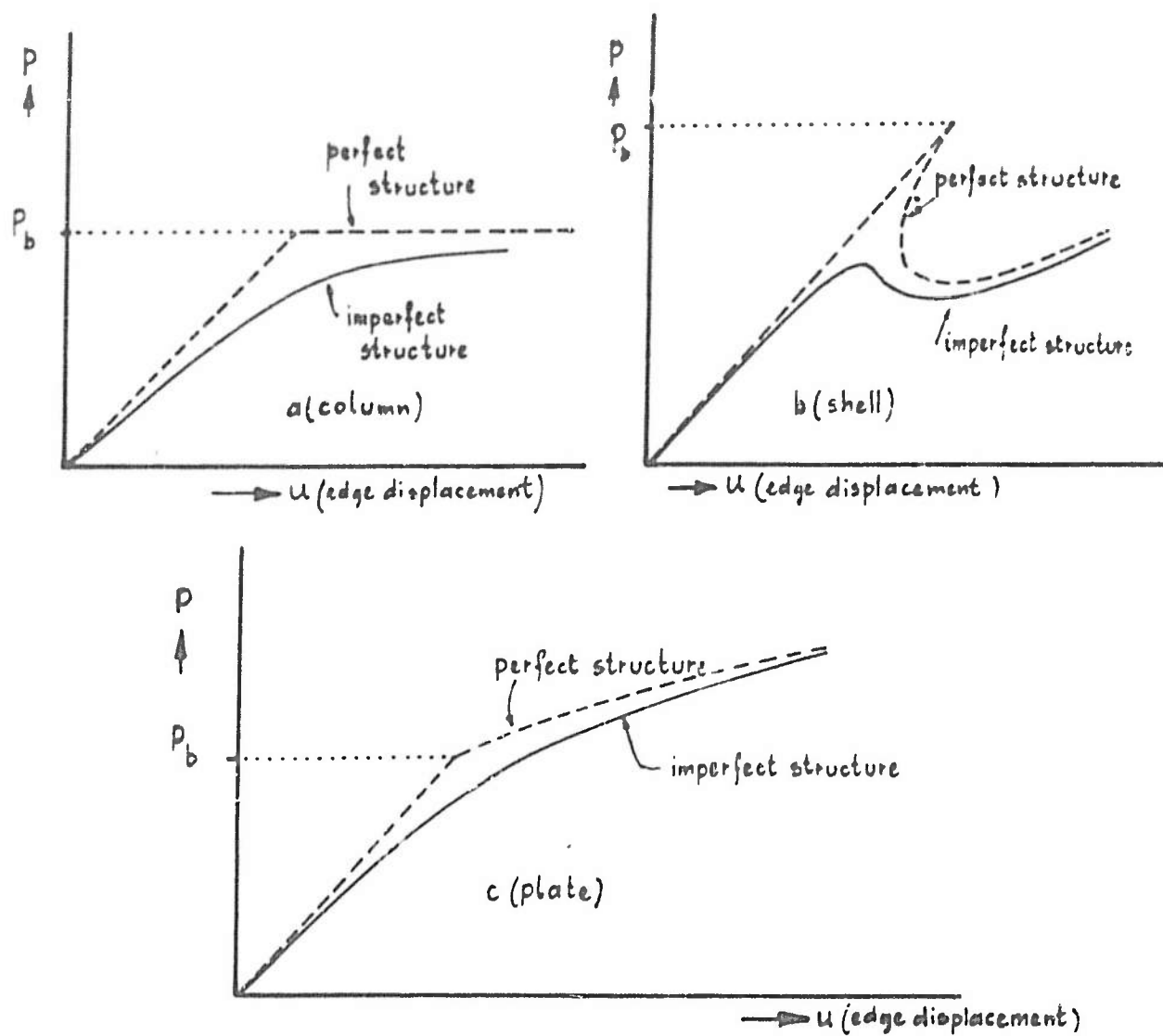


Fig.4 Schematic representation of the effect of initial imperfections

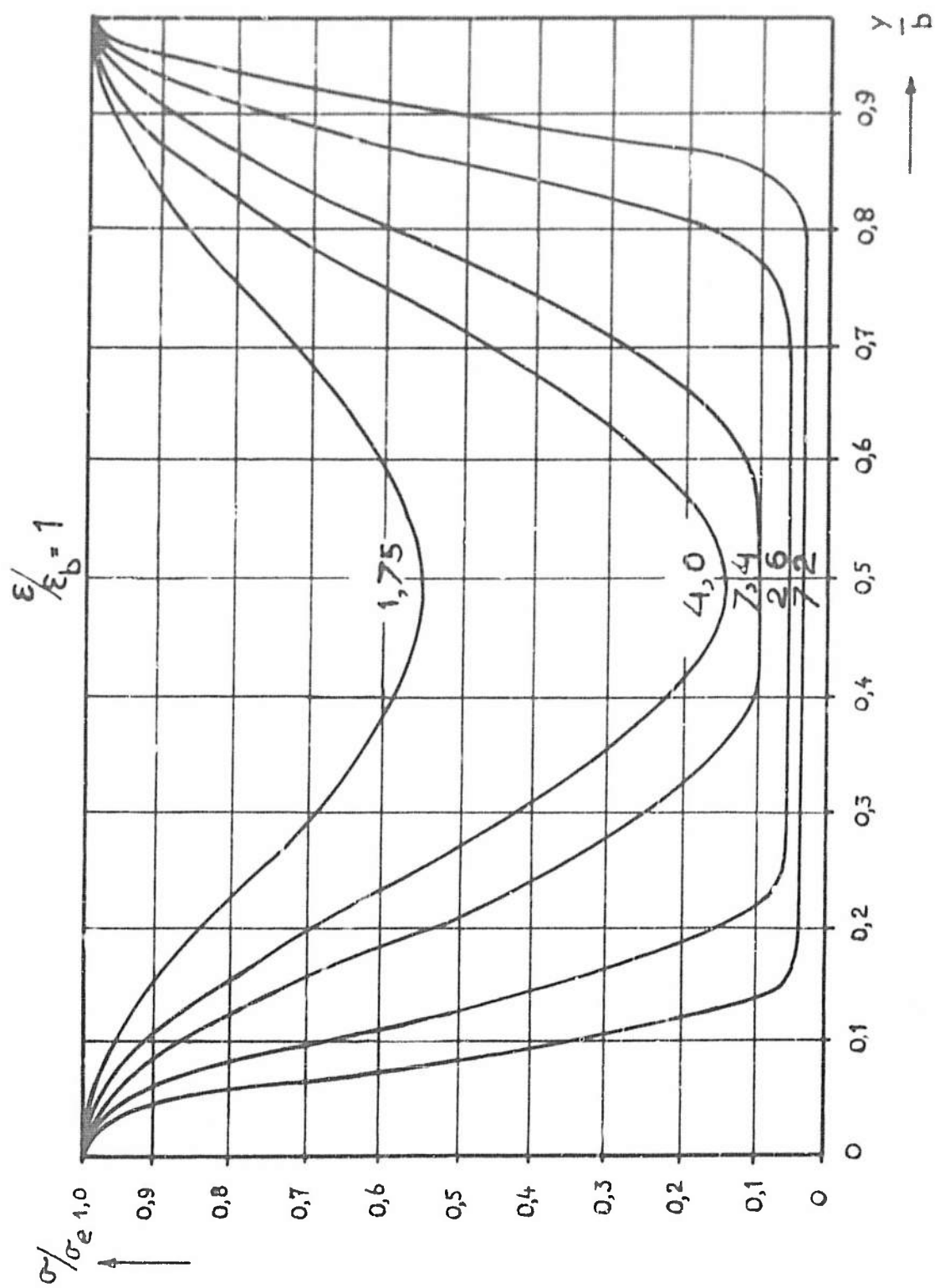
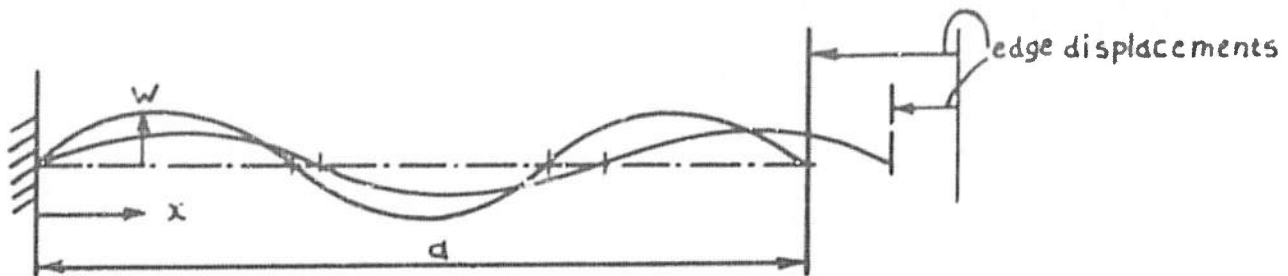


Fig. 5 Relative decrease of membrane stresses with increasing edge displacement beyond buckling. Longitudinal compressive strain ϵ of a long rectangular plate of width b



$$u = \int_0^a \left[1 + \left(\frac{\partial w}{\partial x} \right)^2 \right]^{\frac{1}{2}} dx - a = \frac{1}{2} a \int_0^a \left(\frac{\partial w}{\partial x} \right)^2 dx$$

Fig. 6 Shortening of a plate due to buckling

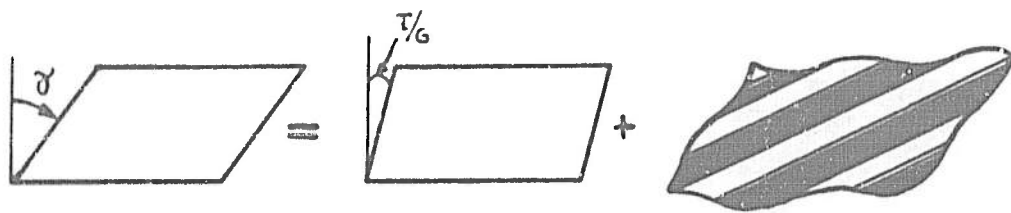


Fig. 7 Shear angle of a rectangular plate following from oblique waves

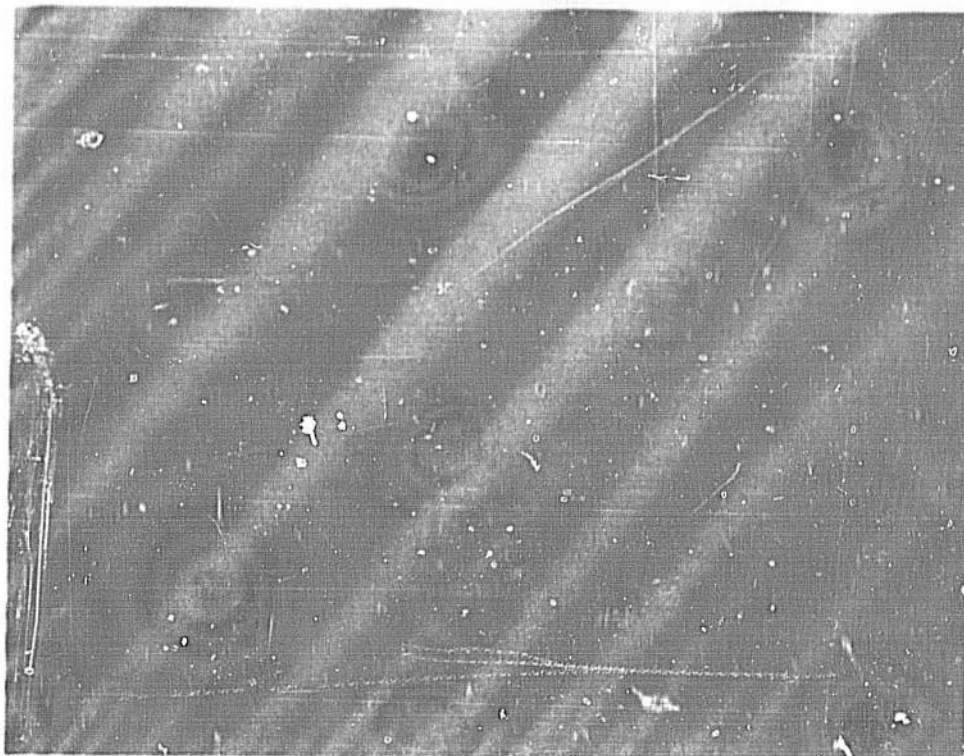


Fig.8 Wave pattern for large strain ratio γ/γ_b

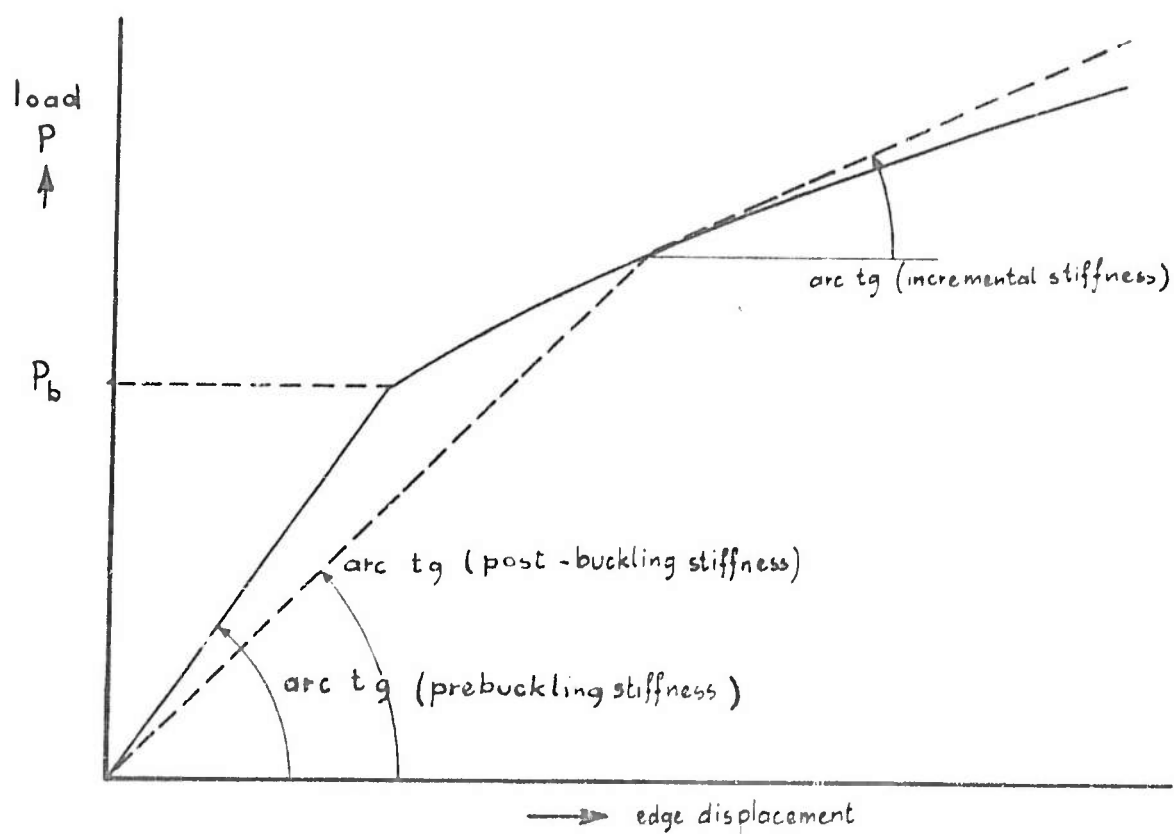


Fig.9 Stiffness of buckled panels

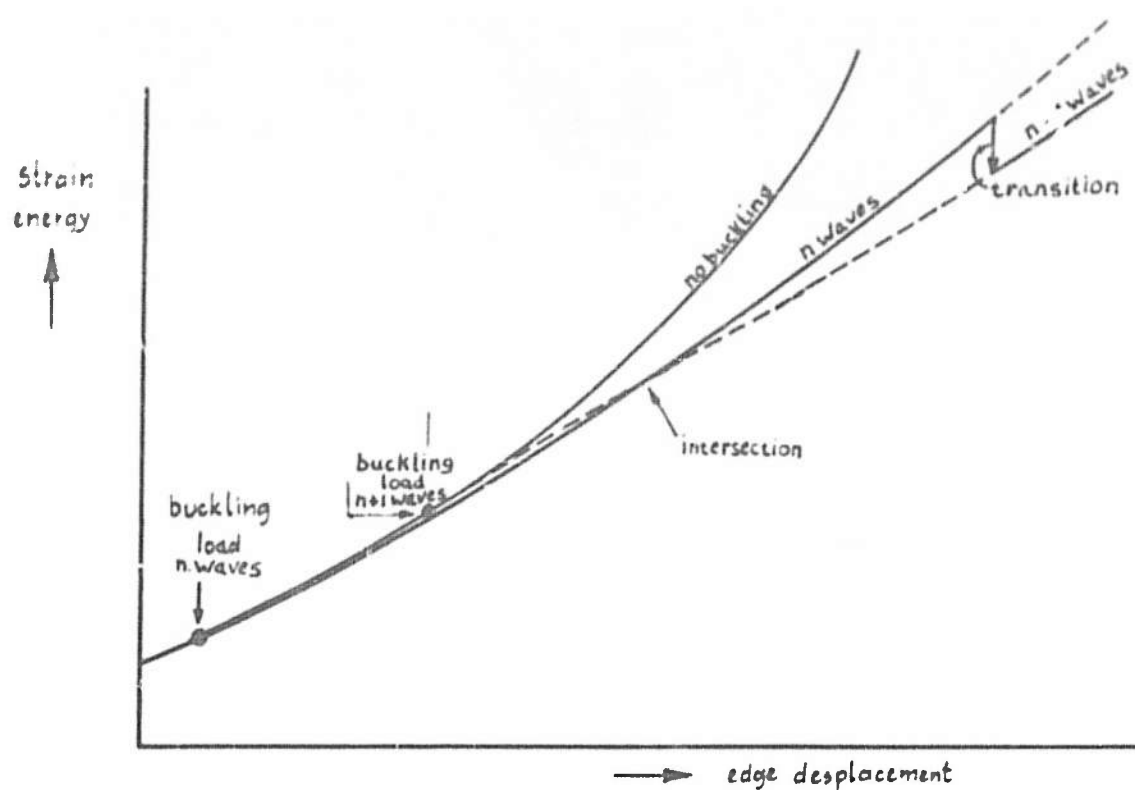


Fig. 10 Explosive change of buckle pattern

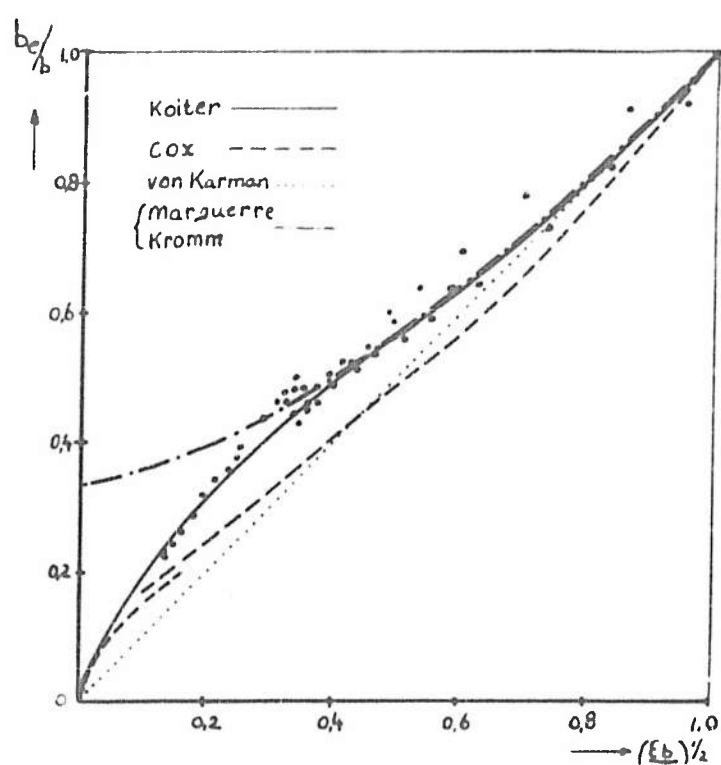


Fig. 11 Effective width

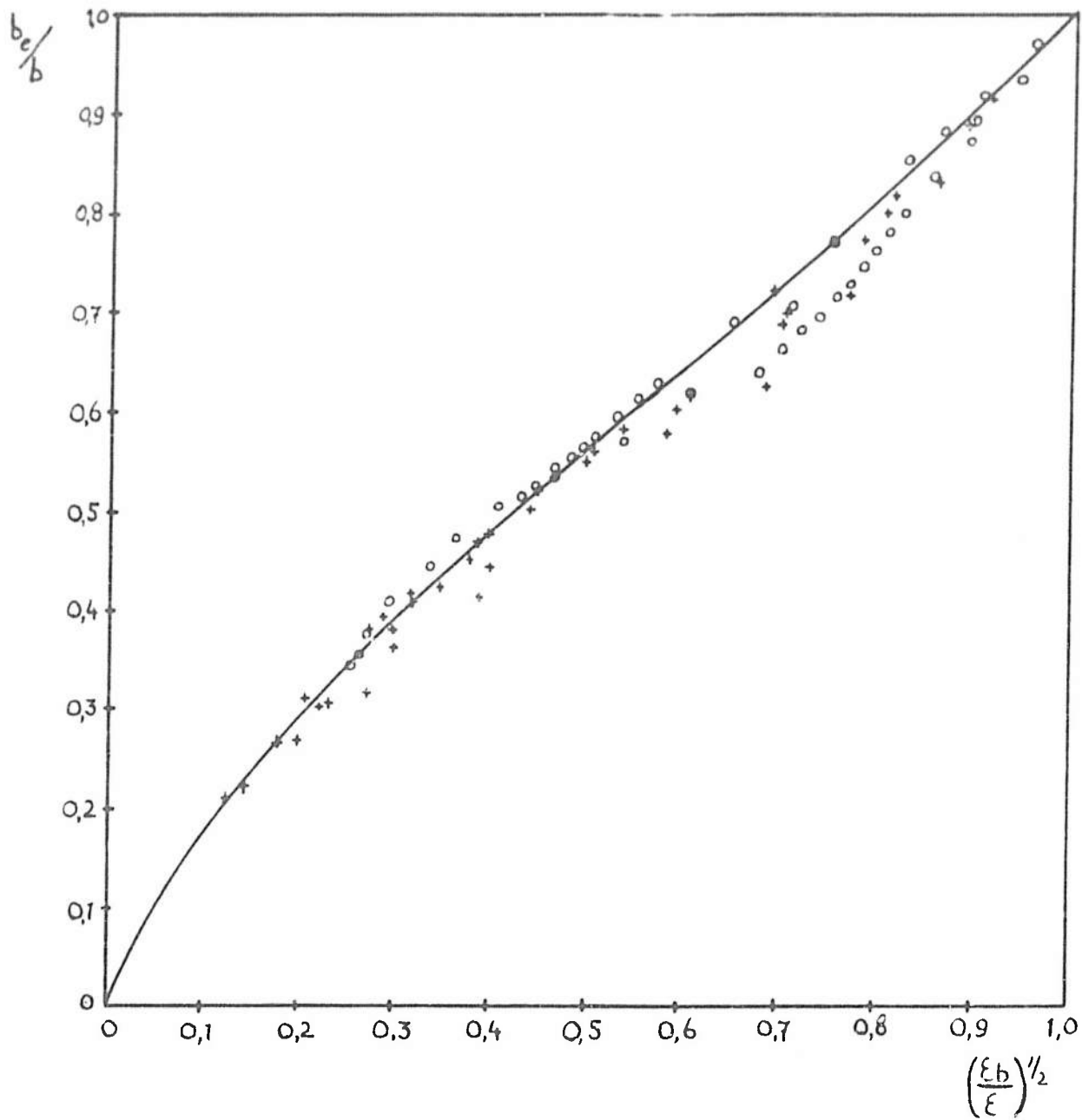


Fig.12 Effective width in the plastic range with 24S-T (+) and 75S-T (o) clad and unclad material

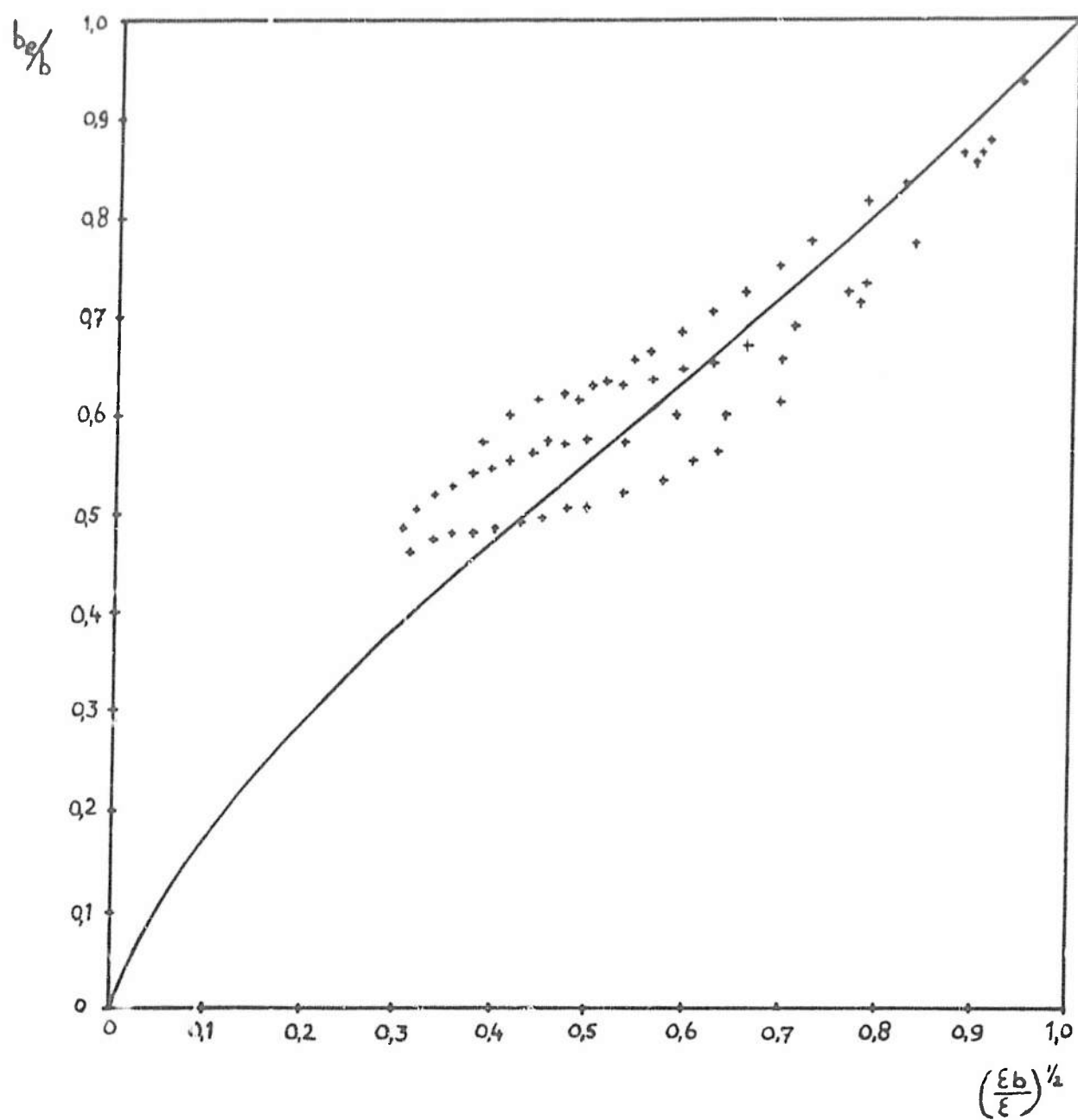


Fig.13 Effective width in the plastic range with 2S- $\frac{1}{2}$ H clad material

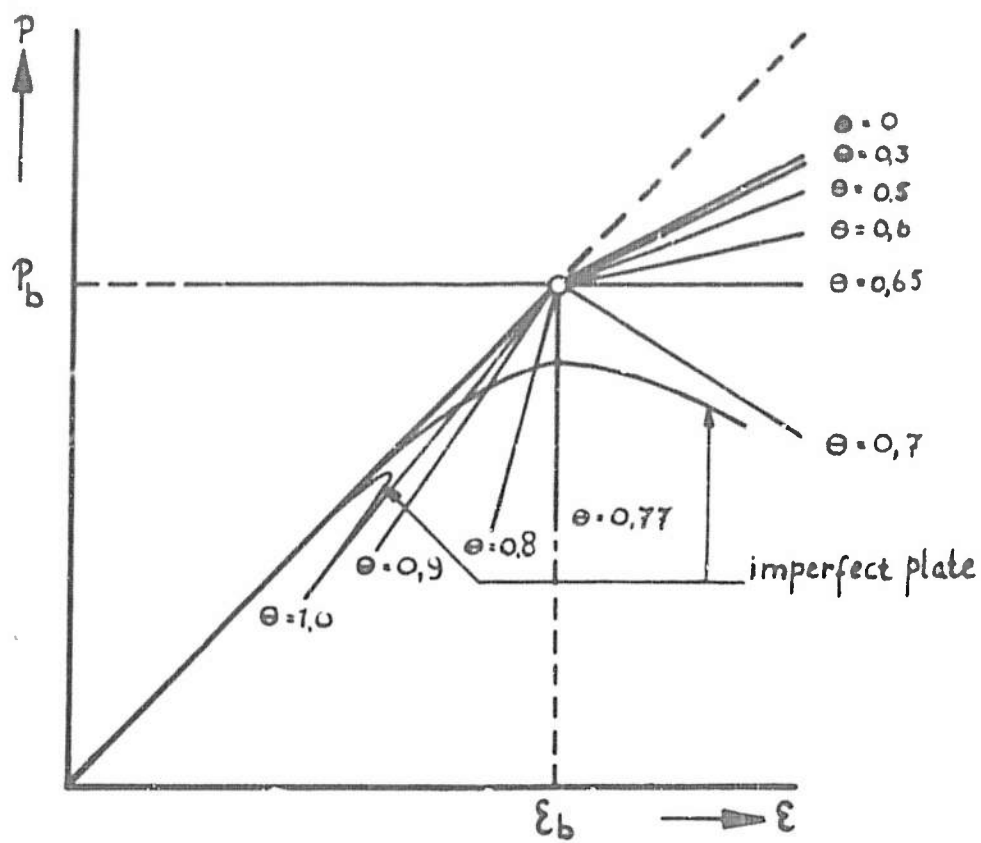


Fig.14 Tangent to post buckling curve at buckling load for cylindrical panels

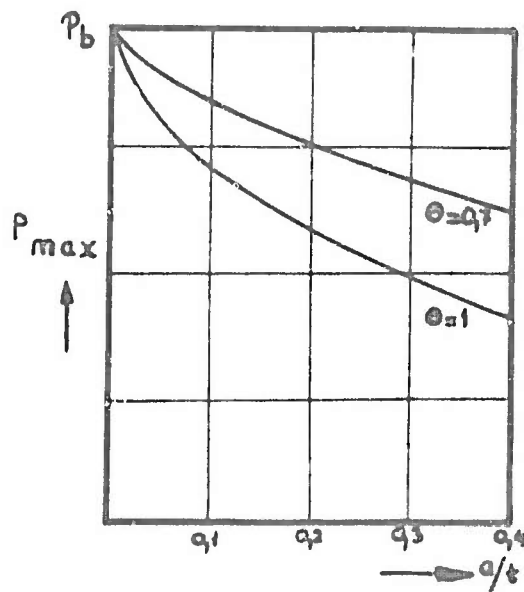


Fig.15 Effect of imperfections on maximum load in the initial post buckling range

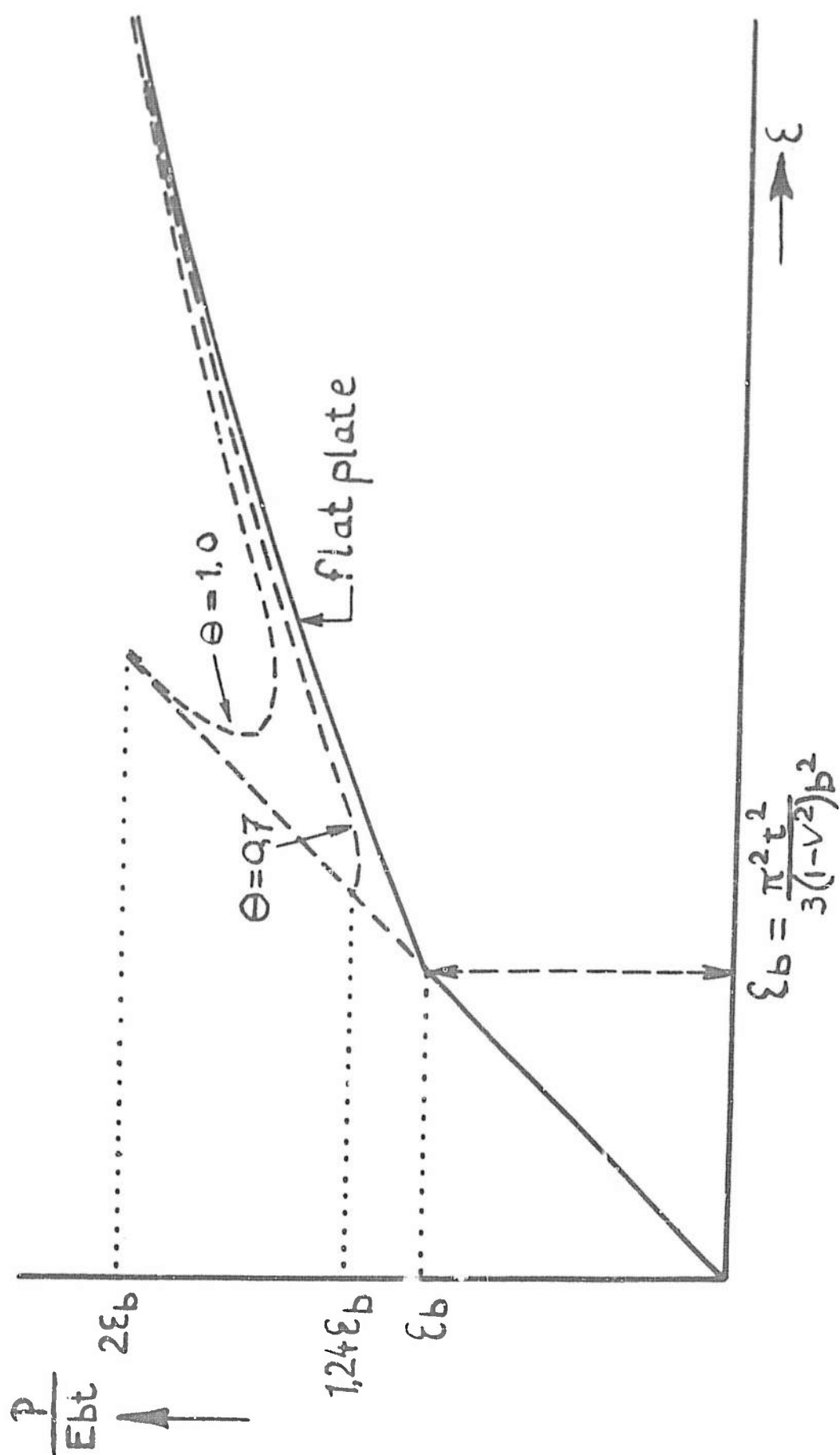


Fig.16 Schematic representation of post-buckling behaviour in advanced stage for narrow cylindrical panels

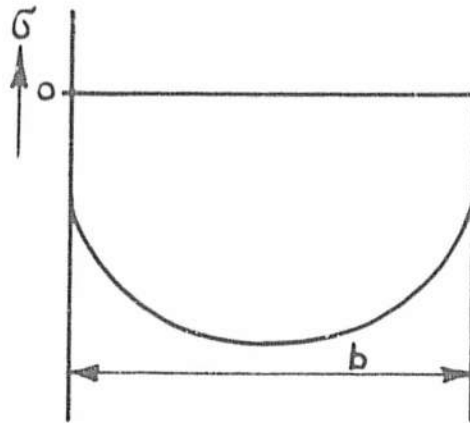


Fig.17 Direct stresses of a skin plate by thermal and mechanical load

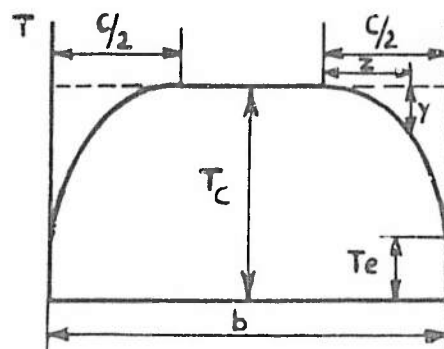


Fig.18 Temperature distribution in the plate

$$y = (T_c - T_e) \left(\frac{2z}{c} \right)^n$$

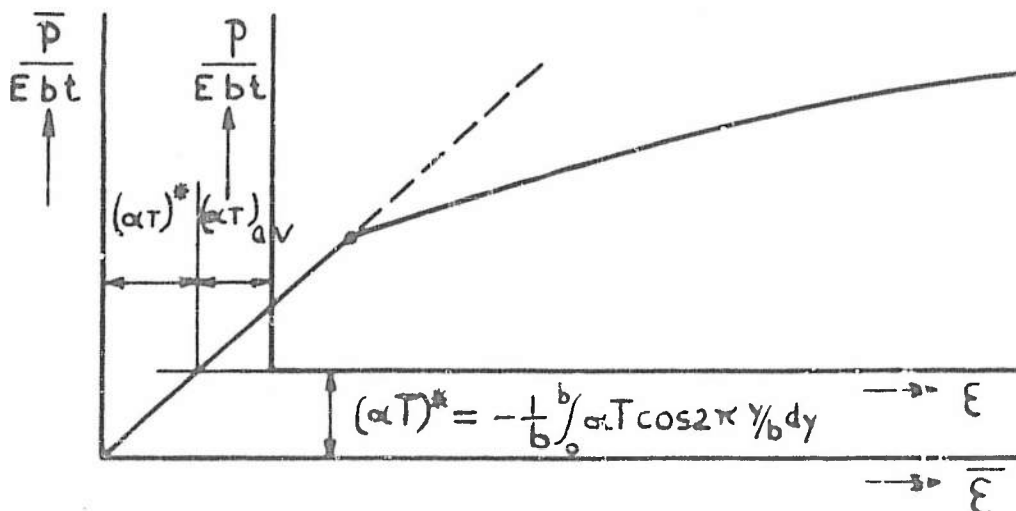


Fig.19 Load strain curve with thermal conditions. \bar{P} , $\bar{\epsilon}$ applies to non-thermal conditions. P , ϵ applies to thermal conditions